**ASSIGNMENT 5**
**DUE THURSDAY MARCH 1**

(1) Let $F_0 = \mathbb{Q}$. For each $n \geq 0$, let $F_{n+1}$ be the field obtained by adjoining to $F_n$ all roots of elements of $F_n$. Let $F = \cup_{n=0}^{\infty} F_n$ (you can view this union taking place in $\overline{\mathbb{Q}}$). Prove that $F$ is a Galois extension of $\mathbb{Q}$. Prove that there are no non-trivial solvable extensions of $F$. Prove that $F$ is not $\overline{\mathbb{Q}}$. Try to give some description of $Gal(F/\mathbb{Q})$.

(2) Let $I$ be a partially ordered set. Let $(G_\alpha)_{\alpha \in I}$ be a collection of groups labelled by the elements of $I$. Assume that we are given group homomorphisms $\phi_{\alpha\beta} : G_\beta \to G_\alpha$ for each pair $\alpha, \beta \in I$ such that $\alpha \leq \beta$. Such data $(I, G_\alpha, \phi_{\alpha,\beta})$ is called an inverse system of groups.

We define the group $\lim_{\leftarrow} G_\alpha$ (called the inverse limit or projective limit) to be the set of all sequences $(g_\alpha)_{\alpha \in I}$ (where $g_\alpha \in G_\alpha$) such that if $\alpha \leq \beta$, then $\phi_{\alpha,\beta}(g_\beta) = g_\alpha$. The group structure is pointwise multiplication of sequences. (So $\lim_{\leftarrow} G_\alpha$ is a subgroup of $\prod_\alpha G_\alpha$).

(a) Formulate and prove a universal property satisfied by $\lim_{\leftarrow} G_\alpha$.

(b) Let $F \subset K$ be an infinite Galois extension. Prove that

$$Gal(K/F) \cong \lim_{\leftarrow} Gal(L/F)$$

where $L$ ranges over intermediate fields $F \subset L \subset K$ such that $L$ is finite and Galois over $F$. Note that you will first have to set up the inverse system. (You just need to prove that this is an isomorphism of groups, but in fact there is a natural topology on an inverse limit and this is actually an isomorphism of topological groups.)

(c) Consider the case where $F = \mathbb{F}_p$ and $K = \overline{\mathbb{F}_p}$. Give a description of the inverse system in this case and its limit. Relate this to the description of $Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ that was given in class.

(3) If $k$ is a finite field, show that every subset of $\mathbb{A}_k^n$ is an affine variety.

(4) Consider $X = Z(xy - z) \subset \mathbb{A}^3$. Show that $X$ is isomorphic to $\mathbb{A}^2$.

(5) A topological space $X$ is called disconnected if $X$ can be written as a disjoint union $X = A \sqcup B$ of two non-empty closed subsets $A, B$. Prove that an affine variety $X$ is disconnected if and only if $\mathcal{O}(X)$ is the direct sum (as rings) of two non-zero ideals.

(6) Consider $k = \mathbb{C}$. We have two topologies on $\mathbb{A}^n$, the Zariski topology and the Euclidean topology, the latter coming from regarding $\mathbb{A}^n$ as $\mathbb{R}^{2n}$. Show that if $X \subset \mathbb{A}^n$ is closed in the Zariski topology, then it is closed in the Euclidean topology. Find a subset of $\mathbb{A}^1$ which is closed in the Euclidean topology which is not closed in the Zariski topology.