Algebra II final

Name:

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1. Find the Galois groups of the polynomial $x^4 - 2$ over each of the fields $\mathbb{Q}, \mathbb{F}_3$, and $\mathbb{F}_5$. You may use without proof the following facts:
   - $x^4 - 2$ is irreducible over $\mathbb{Q}$.
   - $x^4 - 2 = (x^2 - x - 1)(x^2 + x - 1)$ over $\mathbb{F}_3$.
   - $x^4 - 2$ is irreducible over $\mathbb{F}_5$.

2. Let $F \subset K$ be a Galois extension with Galois group G. Suppose that an intermediate field $F \subset E \subset K$ and a subgroup $H \subset G$ correspond, in the sense that $H = Gal(K/E)$. Prove that $F \subset E$ is a Galois extension if and only if H is a normal subgroup of G.

3. Let R be a Noetherian commutative ring and let M be a finitely generated R-module. Suppose that $f : M \to M$ is a surjective R-module morphism. Prove that f is injective. (You may use the following result: if M is a finitely generated module over a Noetherian ring, then there are no infinite ascending chains of submodules of M.)

4. Let V be an irreducible complex representation of a finite group G. Let $H \subset G$ be a subgroup of index k. Let $W \subset V$ be an H-invariant subspace.
   (a) Prove that $\dim W \geq \frac{1}{k} \dim V$.
   (b) Prove that if $\dim W = \frac{1}{k} \dim V$, then W is an irreducible H-representation.

5. Let G be a finite group. Prove that the following are equivalent.

(a) For every $g \in G$, there exists $h \in G$ such that $g^{-1} = hgh^{-1}$.

(b) For every complex representation V of G, $V \cong V^*$.

6. Let k be an algebraically closed field. Recall the following results.

Zariski's Lemma
If $k \subset F$ is a field extension such that F is finitely generated as a k-algebra, then $F = k$.

Weak form of Hilbert's Nullstellensatz
If $I \subsetneq k[x_1, \ldots, x_n]$ is a proper ideal, then $Z(I) \neq \emptyset$.

(a) Prove Zariski's Lemma using the weak form of the Nullstellensatz.

(b) Prove the weak form of Nullstellensatz using Zariski's lemma.