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1 Introduction

The simplest 1-dimensional object that isn't $\mathbb{R}$ is

$$S^1 := \{z \in \mathbb{C} | |z| = 1\} = [0,1]/\sim$$

where $0 \sim 1$.

Consider the 2-sphere S^2 :

$$S^2 = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$$

It can be regarded as

- The level set $F^{-1}(1)$ of $F : \mathbb{R}^3 \to \mathbb{R}$ where $F(x,y,z) = x^2 + y^2 + z^2$
- The Riemann sphere $\hat{C} := \mathbb{C} \cup \{\infty\} = U \coprod V/\sim$ where $U = \{z \in \mathbb{C}\}$ and $V = \{w \in \mathbb{C}\}$ with $z \in U \sim w \in V \leftrightarrow z = w^{-1}$ for $z \neq 0$.
- Stereographic projection defines parametrizations of $S^2 \setminus \{N\}$ and $S^2 \setminus \{S\}$, where $N = (0,0,1)$ and $S = (0,0,-1)$. We define $\pi : S^2 \setminus \{N\} \to \mathbb{R}^2$ by

  $$(u,v) = \pi(x,y,z)$$

  where

  $$x = \frac{4u}{u^2+v^2+4}$$
  $$y = \frac{4v}{u^2+v^2+4}$$
  $$z = \frac{2(u^2+v^2)}{u^2+v^2+4}$$

- $$S^2 = D^2_+ \coprod D^2_-/\sim$$

  where

  $$D^2_+ = \{(x,y,z) | z \geq 0, x^2+y^2+z^2 = 1\}$$
  $$D^2_- = \{(x,y,z) | z \leq 0, x^2+y^2+z^2 = 1\}$$

  and $(x,y,z) \in D^2_+ \sim (x',y',z') \in D^2_- \longleftrightarrow z = 0, x = x', y = y'$

Examples of 2-manifolds:

- The torus: We identify the sides of a square as follows
- The Klein bottle: We identify the sides of a square as follows
- The real projective plane: We identify the sides of a square as follows
- genus g oriented 2-manifold: Identify the edges of a $4g$-gon as follows. All $4g$ vertices are identified to the same point.

Manifolds of dimension n are objects parametrized locally by open sets in $\mathbb{R}^n$. For example, S^2 is parametrized locally by open sets in $\mathbb{R}^2$.

Themes:

1. Smoothness: Some level sets $F^{-1}(a_1, \ldots, a_m)$ (for $(a_1, \ldots, a_m) \in \mathbb{R}^m$ and$F : \mathbb{R}^N \to \mathbb{R}^m$) are not smooth. We will establish a criterion for when a level set is a smooth manifold.

   For example, $F : \mathbb{C}^2 \to \mathbb{C}$ defomed bu $F(z, w) = z^3 - w^2$. In this case, $F^{-1}(0)$ is not locally modelled on $\mathbb{R}^2$. We see this by examining $F^{-1}(0) \cap \mathbb{R}^2 = \{(x, y) \in \mathbb{R}^2 | x^3 = y^2\}$. This is $y = \pm x^{3/2}$ which is not smooth at $(0, 0)$.

2. Tangent space T_xM to a manifold M at $x \in M$

   - If $F : \mathbb{R}^n \to \mathbb{R}^N$ and
     $$(dF)_{ij} = \frac{\partial F_i}{\partial x_j}$$
     ($i = 1, \ldots, N$) and ($j = 1, \ldots, n$)) then $\mathrm{Im}(dF)_x$ is the tangent space to $M := F(\mathbb{R}^n)$ at $F(x)$.
   - If we look at $b = F^{-1}(a)$ for $F : \mathbb{R}^n \to \mathbb{R}^N$, then the tangent space to $F^{-1}(a)$ at y (where $F(y) = a$) is the kernel of dF:
     $$\{\xi \in \mathbb{R}^n | (dF)_y(\xi) = 0\}$$
     We will make sense of manifolds and their tangent spaces in a way that is independent of their description as subsets of $\mathbb{R}^N$.

   Remark: In fact all manifolds of dimension n can be written as subsets of $\mathbb{R}^{2n}$ (Whitney embedding theorem)

3. Tangent bundles: The set of points in a manifold M together with their tangent spaces gives an object called a *tangent bundle* of M. For example, for $S^1 \subset \mathbb{R}^2$ the tangent bundle is
   $$(x, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2 || x| = 1, < \xi, x >= 0\}.$$

Tangent bundles of smooth manifolds are also smooth manifolds. The key map is the projection $\pi : TM \to M$ with $\pi^{-1}(x) = T_xM$. Locally $TM|_U \cong U \times \mathbb{R}^m$ but usually we don't have $TM \cong M \times \mathbb{R}^m$.

4. Vector bundles over a manifold M generalize the tangent bundle TM of M. These are objects E with a surjective map $\pi : E \to M$ for which $\pi^{-1}(x)$ has the structure of a vector space.

5. Sections: A section of the tangent bundle consists of the specification of a tangent vector at each $x \in M$, in other words $s : TM \to M$ with $\pi \circ s = \text{id}$. A section of the tangent bundle is called a vector field.

   Do sections exist, must they have a zero somewhere? We will prove a theorem that on S^2 there is no nowhere vanishing section.

6. Integration on manifolds: Recall the change of variables formula

$$\int_{g(U)} f(y)dy = \int_U f \circ g(x)|dg/dx|dx$$

   where $g : \mathbb{R} \to \mathbb{R}$, U open in $\mathbb{R}$. More generally $y = (y_1, \ldots, y_n)$, $x = (x_1, \ldots, x_n)$

$$\int_{g(U)} f(y)dy = \int_U f \circ g(x)|\text{det} dg_x|dx$$

   Integration of *functions* on manifolds is not well defined. We must pass to *differential forms*, whose description in terms of a local parametrization of the manifold transforms under change of parametrization so that integration of differential forms is well defined.

7. Orientability: Orientability of a manifold M is a consistency condition on parametrization of tangent spaces. It is equivalent to the existence of a nowhere vanishing n-form (if n is the dimension of M).

   The sphere, the torus and the manifolds of dimension $4g$ obtained by gluing the sides of a $(4g)$-gon are orientable. Examples of nonorientable manifolds include the Klein bottle, the Möbius strip and the real projecctive space (obtained by attaching the boundary of a disk to the boundary of a Möbius strip). Integration of differential forms can only be defined on orientable manifolds.

8. Differential calculus of forms: We define the exterior derivative d which takes r-forms to $(r + 1)$-forms.

9. Stokes' Theorem
10. De Rham cohomology

2 Smooth functions

Definition 2.1 *A function $F : \mathbb{R}^m \to \mathbb{R}^n$ is* smooth *(or C^∞) if its partial derivatives to all orders exist and are continuous.*

Definition 2.2 *The Jacobian $(dF)_x$ is the matrix*

$$\frac{\partial F_i}{\partial x_j} \quad (i = 1, \ldots, n;\ j = 1, \ldots, m)$$

Definition 2.3 *F is a* homeomorphism *if it is a continuous bijective map whose inverse is also continuous.*

Definition 2.4 *F is a* diffeomorphism *if it is a smooth homeomorphism whose inverse is also smooth.*

Theorem 2.5 (Chain Rule) *Suppose $F : \mathbb{R}^m \to \mathbb{R}^n$ and $G : \mathbb{R}^p \to \mathbb{R}^m$. Then $d(F \circ G)_x = (dF)_{G(x)} \circ (dG)_x$.*

Definition 2.6 *A* topological manifold *M of dimension m is a topological space which is Hausdorff and second countable (i.e. there is a countable base of open sets for its topology) and for which each point has an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^m$.*

Definition 2.7 *A* chart *of M is (U, ϕ) where U is an open subset of M and $\phi : U \to \mathbb{R}^m$ is a homeomorphism.*

Definition 2.8 *Let $\pi_k : \mathbb{R}^m \to \mathbb{R}$ be projection onto the k-th coordinate. Let $x_k = \pi_k \circ \phi : U \to \mathbb{R}$. The x_k are* coordinate functions *of the chart.*

Definition 2.9 *Let (U_1, ϕ_1) and (U_2, ϕ_2) be charts of M. They are C^∞-compatible if $\phi_1 \circ \phi_2^{-1}$ and $\phi_2 \circ \phi_1^{-1}$ are C^∞-mappings whenever they are defined, in other words whenever $\phi_1 \circ \phi_2^{-1}$ is a bijective map from $\phi_2(U_1 \cap U_2)$ to $\phi_1(U_1 \cap U_2)$).*

Definition 2.10 *An* atlas *for M is a collection $\{V_\alpha, \phi_\alpha)$ with $\cup_\alpha V_\alpha = M$.*

Definition 2.11 *A C^∞-atlas is an atlas for which all the charts are C^∞-compatible.*

Definition 2.12 *A C^∞-*structure *is a maximal C^∞-atlas (every chart which is compatible with every chart of the atlas is already a member of it).*

Definition 2.13 *A* C^∞-manifold *is a topological manifold equipped with a* C^∞ *atlas.*

Remark 2.14 *There may be inequivalent* C^∞*-structures on a topological manifold – manifolds homeomorphic but not diffeomorphic – for example* $\mathbb{R}^4$ *(S. Donaldson) and* S^7 *(J. Milnor).*

Example 2.15 *1.* $\mathbb{R}^n$*, with the chart* $\phi =$ *identity*

2. $S^1 = \{e^{i\theta} \in \mathbb{C}\}$ *Take* $V_1 = \{e^{i\theta} : -\epsilon < \theta < \pi + \epsilon\}$ *and* $V_2 = \{e^{i\theta} : \pi < \theta < 2\pi\}$ $\phi_1 : V_1 \to (-\epsilon, \pi + \epsilon)$ *and* $\phi_2 : V_2 \to (\pi, 2\pi)$ *given by* $\phi_i(e^{i\theta}) = \theta$ *for* $i = 1, 2$*. Then* $\phi_1(V_1 \cap V_2) = (-\epsilon, 0) \coprod (\pi, \pi + \epsilon)$*.* $\phi_2 \circ \phi_1^{-1}|_{(-\epsilon,0)}(\theta) = \theta + 2\pi$ *while* $\phi_2 \circ \phi_1^{-1}|_{(\pi,\pi+\epsilon)}(\theta) = \theta$*.*

3. $S^n = \{(x_0, \ldots, x_n) : \sum_i x_i^2 = 1\}$ *Charts* $U_i^\pm = \{\pm x_i > 0\}(i = 0, \ldots, n)$ *Chart maps* $\phi_i^\pm : U_i^\pm \to \mathbb{R}^n$
$$\phi_i(x_0, \ldots, x_n) = (x_0, \ldots, \hat{x}_i, \ldots, x_n)$$
So
$$\phi_j^+ \circ (\phi_i^+)^{-1}(z_1, \ldots, z_n) = (z_1, \ldots, \hat{z}_j, \ldots, \sqrt{1 - \sum_k z_k^2}, \ldots, z_n)$$
(where the $\sqrt{1 - \sum_k z_k^2}$ *occurs in the* $(i-1)$*-th place).*

4. *Stereographic projection on* S^2*: There are two systems of coordinates on* S^2*,* (u, v) *and* $(\hat{u}, \hat{v})$*.*
$$\hat{u} = \frac{u}{u^2 + v^2}$$
and
$$\hat{v} = \frac{-v}{u^2 + v^2}$$
and $u(x, y, z) = \frac{x}{1-z}$ *and* $v(x, y, z) = \frac{y}{1-z}$*. The map* $\phi_N : S^2 \setminus \{(0, 0, 1)\} \to \mathbb{R}^2$ *(stereographic projection from* N *on the plane through the equator) is*
$$\phi_S = (\hat{u}, \hat{v})$$
where
$$\hat{u} = \frac{x}{1 + z}$$
and
$$\hat{v} = \frac{y}{1 + z}$$
So
$$\frac{u}{u^2 + v^2} = \frac{x/(1-z)}{x^2/(1-z)^2 + y^2/(1-z)^2}$$

$$= \frac{x(1-z)}{x^2+y^2} = \frac{x}{1+z} = \hat{u}.$$

Likewise

$$\frac{-v}{u^2+v^2} = \hat{v}.$$

More generally this method in fact works for S^n:

$$\phi_N(\bar{x}, x_{n+1}) = \bar{y}$$

where

$$\bar{y} = \frac{\bar{x}}{1-x_{n+1}}.$$

Also, $\phi_S(\bar{x}, x_{n+1}) = \frac{\bar{x}}{1+x_{n+1}}$.

5. *Real projective space $\mathbb{R}P^n = S^n/\sim$ where $x \sim -x$ (x and $-x$ are related by the antipodal map, which is multiplication by -1). Denote by $[x_0 : \ldots : x_n]$ the equivalence class of $(x_0, \ldots, x_n)$ under the antipodal map*

$$U_k = \{[x] \in \mathbb{R}P^n : x_k \neq 0\}$$

and

$$[x_0 : \ldots, x_n] \mapsto \operatorname{sgn}(x_k)(x_0, \ldots, \hat{x_k}, \ldots, x_n)$$

We may check that this endows $\mathbb{R}P^n$ with the structure of a C^∞ atlas.

6. *$\mathbb{R}P^2 = D^2/\sim$ where $s \sim -s$ for $s \in \partial D^2$ (the elements in the boundary of D^2).*

7. *Complex projective space $\mathbb{C}P^n$*

$$\mathbb{C}P^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\sim$$

where $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$ for $\lambda \in \mathbb{C} \setminus \{0\}$. The equivalence class is normally denoted $[z_0 : \ldots : z_n]$. The sets

$$U_i = \{[z_0 : \cdots : z_n] | z_i \neq 0\}$$

form a covering of $\mathbb{C}P^n$.

Define

$$\phi_i : [z_0 : \ldots z_n] \mapsto (\frac{z_1}{z_i}, \ldots, \hat{z_i}, \ldots, \frac{z_n}{z_i}).$$

$$\phi_i : U_i \to \mathbb{C}^n \cong \mathbb{R}^{2n}$$

$$\phi_i^{-1} : (w_1, \ldots, w_n) \mapsto [w_1, \ldots, 1, \ldots, w_n]$$

so

$$\phi_j \circ \phi_i^{-1} : (w_1, \ldots, w_n) \mapsto (\frac{w_1}{w_j}, \ldots, \frac{1}{w_j} \ldots, \hat{w_j}, \ldots, \hat{w_n}, \ldots, w_j).$$

Remark 2.16 $\mathbb{C}P^1 = S^2$

$$\mathbb{C}P^1 = \{[z_1 : z_2]\} = \{[1 : z]\} \cup \{[w : 1]\} / \sim$$

Here $[w : 1] \sim [1 : 1/w]$ *if* $w \neq 0$.

3 The inverse function theorem

Theorem 3.1 *(Inverse function theorem)*

Let A be an open subset of $\mathbb{R}^n$ and let $F : A \to \mathbb{R}^n$ be C^1 in an open neighbourhood containing $\bar{a}$. Suppose the square matrix $dF_{\bar{a}}$ is invertible. Then there is an open neighbourhood V of $\bar{a}$ and an open set W containing $F(V)$ such that $F : V \to W$ has an inverse $F^{-1} : W \to V$ which is continuous and differentiable , and $(dF_{\bar{x}})^{-1} = d(F^{-1}_{\bar{y}})$ if $F(\bar{x}) = \bar{y}$.

Proof: See Apostol, *Mathematical Analysis* (2nd edition) Chaps. 13.2 and 13.3 or Spivak, *Calculus on Manifolds* Thm. 2.11.

Definition 3.2 *(Rank of a matrix) Rank* $(F|_a) = \dim\left(\operatorname{Im}(dF)_a\right)$.

Theorem 3.3 *(Constant rank theorem) Suppose $U \subset \mathbb{R}^n$, $F = (f_1, \ldots, f_n) : U \to \mathbb{R}^m$ is C^∞ in a neighbourhood of $\bar{a}$, and $\operatorname{Rk}(F)_x = r$ for all x in a neighbourhood of $\bar{a}$. Then there are open neighbourhoods U (resp. V) of $\bar{a}$ (resp. $F(\bar{a})$) and diffeomorphisms $\phi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^m$ with*

$$\begin{array}{ccc} U & \xrightarrow{F} & V \\ \phi \downarrow & & \downarrow \psi \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \end{array}$$

such that $\psi \circ F \circ \phi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_r, 0, \ldots, 0)$.

Proof: WLOG $\bar{a} = 0$ and $F(\bar{a}) = 0$ (by composing with translations). WLOG

$$(dF)_{\bar{a}} = \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}$$

(by composing with suitable linear transformations of $\mathbb{R}^n$ resp. $\mathbb{R}^m$).

Define $\phi(x_1, \ldots, x_n) = (f_1(x), \ldots, f_r(x), x_{r+1}, \ldots, x_n)$ where r is the rank of f. Then

$$(d\phi)_0 = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_r} & ? \\ \vdots & & \vdots & \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_r} & ? \\ 0 & 0 & 0 & 1_{n-r} \end{pmatrix}$$

So F is invertible if and only if there is a local neighbourhood V of 0 in $\mathbb{R}^n$ such that $\phi|_V$ is a C^∞-diffeomorphism from V to $\phi(V)$. Define $g = F \circ \phi^{-1} : \phi(V) \to \mathbb{R}^m$ for $z = (z_1, \dots, z_n) \in \phi(V)$.

$$g(z) = (z_1, \dots, z_r, g_{r+1}(z), \dots, g_m(z))$$

since ϕ and F agree on the first r components. Hence

$$(dg)_z = \begin{pmatrix} 1_r & 0 & 0 & 0 \\ ? & \frac{\partial g_{r+1}}{\partial z_{r+1}} & \dots & \frac{\partial g_{r+1}}{\partial z_m} \\ \vdots & \vdots & \vdots & \vdots \\ ? & \frac{\partial g_m}{\partial z_{r+1}} & \dots & \frac{\partial g_m}{\partial z_m} \end{pmatrix}$$

Since F (and hence also g) has constant rank r on a neighbourhood of 0,

$$\frac{\partial(g_{r+1}, \dots, g_m)}{\partial(z_{r+1}, \dots, z_m)} = 0. \tag{1}$$

Hence each of the $g_{r+1}, \dots, g_m$ depends only on $z_1, \dots, z_r$ in a neighbourhood of 0 (by the Mean Value Theorem applied to the last $m - r$ coordinates). Recall that the Mean Value Theorem says that if $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ and $[a, b] \subset U$ is a line segment then

$$||(b) - f(a)|| \leq ||b - a|| \sup_{x \in [a,b]} ||f'(x)||$$

In our situation, $\sup_{x \in [a,b]} f'(x) = 0$ for any $[a, b]$ because of (1). Define

$$\Psi(y_1, \dots, y_m) = (y_1, \dots, y_r, y_{r+1} - g_{r+1}(y_1, \dots, y_r, 0, \dots, 0) \dots, y_m - g_m(y_1, \dots, y_r, 0, \dots, 0).$$

Hence on a neighbourhood of 0 in $\mathbb{R}^n$,

$$\Psi \circ F \circ \phi^{-1}(z_1, \dots, z_r) = \Psi \circ g(\bar{z}) = \Psi(z_1, \dots, z_r, g_{r+1}(\bar{z}), \dots, g_m(\bar{z}))$$

$$= (z_1, \dots, z_r, g_{r+1}(\bar{z}) - g_{r+1}(z_1, \dots, z_r, 0, \dots, 0), \dots, g_m(\bar{z}) - g_m(z_1, \dots, z_r, 0, \dots, 0)$$
$$= (z_1, \dots, z_r, 0, \dots, 0)$$

as $g_{r+1}, \dots, g_m$ depend only on $z_1, \dots, z_r$.

Lemma 3.4 *The rank satisfies*

$$\mathrm{rk}(dF)_x \geq \mathrm{rk}(dF)_a$$

for all x in a neighbourhood of a.

This is because the set of points $x \in \mathbb{R}^n$ where at least one minor of the matrix df_x is nonzero is an open set.

Special cases:

1. **Local submersion theorem:** Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ and $(df)_a$ is onto (for $a \in \mathbb{R}^n$). This implies the rank of $(df)_x$ is m on a neighbourhood of a. Then there are neighbourhoods U, V and maps Ψ and ϕ such that

$$\Psi \circ f \circ \phi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_m).$$

2. **Local immersion theorem:** Suppose $f : \mathbb{R}^n \to \mathbb{R}^{n+m}$ and $(df)_a$ is injective (this implies the rank of $(df)_x$ is n on a neighbourhood of a). Then there are neighbourhoods U, V and maps Ψ and ϕ such that

$$\Psi \circ f \circ \phi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0).$$

3. **Implicit function theorem:** Suppose $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is C^∞. If $f(\bar{x}, \bar{y}) = 0$ (for $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$) and $\det M \neq 0$ (where M is the $m \times m$ matrix $M_{ij} = \frac{\partial f_j}{\partial y_i}$) then for some open neighbourhoods $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ there is a C^∞ function $g : U \to V$ such that

$$f(\bar{x}, \bar{y}) = 0 \leftrightarrow \bar{y} = g(\bar{x})$$

in other words

$$\{(\bar{x}, \bar{y}) | f(\bar{x}, \bar{y}) = 0\}$$

is locally the graph of g.

Proof: (of Implicit Function Theorem) The result follows from the Inverse Function Theorem. Define $F(\bar{x}, \bar{y}) = (\bar{x}, f(\bar{x}, \bar{y}))$. Then $\det(dF)_{\bar{x},\bar{y}} \neq 0$ (here $dF_{\bar{x},\bar{y}}$ is an $(n+m) \times (n+m)$ matrix). By the Inverse Function Theorem, F has a C^∞ inverse $h : W \to A \times B$ for some neighbourhood W of 0 in $\mathbb{R}^{n+m}$ and a neighbourhood A of 0 in $\mathbb{R}^n$ together with a neighbourhood B of 0 in $\mathbb{R}^m$, for which $h(\bar{x}, \bar{y}) = (\bar{x}, k(\bar{x}, \bar{y}))$ for some smooth function k whose domain is an open neighbourhood of 0 in $\mathbb{R}^{n+m}$ and whose range is an open neighbourhood of 0 in $\mathbb{R}^m$. Put $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ with $\pi(\bar{x}, \bar{y}) = \bar{y}$ so $\pi \circ F = f$. Then

$$f\left(\bar{x}, k(\bar{x}, \bar{y})\right) = f \circ h(\bar{x}, \bar{y}) = \pi \circ F \circ h(\bar{x}, \bar{y}) = \bar{y}.$$

For $\bar{x} \in \mathbb{R}^n$ and $\bar{z} \in \mathbb{R}^m$, $f(\bar{x}, \bar{z}) = 0$ implies $\bar{z} = k(\bar{x}, 0)$ as $(\bar{x}, \bar{y}) \mapsto (\bar{x}, k(\bar{x}, \bar{y}))$ is bijective and $f\big(\bar{x}, k(\bar{x}, \bar{y})\big) = 0$ implies $\bar{y} = 0$.

4 Smooth maps between smooth manifolds

Definition 4.1 *Suppose M and N are smooth manifolds. A map $F : M \to N$ is smooth iff for all charts (U, ϕ) for M and (V, ψ) for N, we have the following commutative diagram:*

$$\begin{array}{ccc} U & \xrightarrow{F} & V \\ \downarrow \phi & & \downarrow \psi \\ \phi(U) & \xrightarrow{G} & \psi(V). \end{array}$$

In the above diagram, $G := \Psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \Psi(V)$.

Special case: $F : M \to \mathbb{R}$ is smooth iff $F \circ \phi^{-1}$ is smooth.

Remark 4.2 *To establish that F is smooth it suffices to check for* one *choice of charts U, V with $a \in U$, $F(a) \in V$ because composition of C^∞ maps is C^∞, and*

$$\Psi \circ F \circ \phi^{-1} = \Psi \circ F \circ (\phi')^{-1} \circ (\phi' \circ \phi^{-1})$$

if $\phi \circ (\phi')^{-1}$ is C^∞ (since the charts are C^∞-compatible) so $\Psi \circ F \circ (\phi)^{-1}$ is smooth iff $\Psi \circ F \circ (\phi')^{-1}$ is smooth. Similarly if one replaces Ψ by Ψ', $\Psi \circ F \circ (\phi)^{-1}$ is smooth iff $\Psi' \circ F \circ (\phi)^{-1}$ is smooth.

Remark 4.3 *It is also useful to know that the composition of smooth functions is smooth in order to prove that specific functions are smooth. We can often reduce to specific examples:*

1. *linear functions*
2. *polynomial functions*
3. *roots $x \mapsto x^{1/p}$*
4. *trigonometric functions* $\sin, \cos$
5. *exponential functions*

Definition 4.4
- *F is an immersion at $p \in M$ if $(dG)(\phi)$ is an injective map $\mathbb{R}^m \to \mathbb{R}^n$ (since the previous condition implies* $\mathrm{rank}(dG)(\phi(x)) = m$*).*
- *F is a submersion at $p \in M$ if $(dG)(\phi(p))$ is surjective (this implies rank$(dG)(\phi(x)) = n$)).*

In this situation $m \geq n$.

The following two theorems are special cases of the constant rank theorem:

Theorem 4.5 *(Local immersion theorem) If F is an immersion at p then there exists a chart (U, ϕ) around p and (V, Ψ) around $F(p)$, with $\phi(p) = 0$ and $\Psi(F(p)) = 0$, for which G is the immersion $i : \mathbb{R}^m \to \mathbb{R}^n$ given by $i(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$.*

$$\begin{array}{ccc} U & \xrightarrow{F} & V \\ \downarrow \phi & & \downarrow \psi \\ \phi(U) & \xrightarrow{G} & \psi(V) \end{array}$$

Theorem 4.6 *(Local submersion theorem) If F is a submersion at p then there exist charts (U, ϕ) and (V, Ψ) as above for which G is the projection π where $\pi : \mathbb{R}^m \to \mathbb{R}^n$ given by $\pi : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_n)$.*

Methods to construct manifolds

1. An open set of a manifold is also a manifold
2. If M and N are manifolds then so is $M \times N$
3. The *regular value theorem* (a consequence of the local submersion theorem) yields many examples of manifolds

Definition 4.7 *If $F : \mathbb{R}^m \to \mathbb{R}^n$ is smooth, then $b \in \mathbb{R}^n$ is a* regular value *for F if $\forall a \in F^{-1}(b)$, $(dF)_a$ is a surjective map from $\mathbb{R}^m$ to $\mathbb{R}^n$.*

Definition 4.8 *Suppose* $\dim(M) = m$ *and* $\dim(N) = n$. *Then a point $b \in N$ is a* regular value *for G if for all $a \in F^{-1}(b)$ there are charts (U, ϕ) near a and (V, Ψ) near $F(a)$ for which $(dG)_{\phi(a)}$ is a surjective linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$.*

Recall the commutative diagram

$$\begin{array}{ccc} U \subset M & \xrightarrow{F} & V \subset N \\ \downarrow \phi & & \downarrow \psi \\ \phi(U) & \xrightarrow{G} & \psi(V) \end{array}$$

for ϕ and ψ as above.

Theorem 4.9 *(Regular value theorem) If $b \in N$ is a regular value for F, then $F^{-1}(b)$ is a manifold of dimension $m-n$.*

Proof: Informally: If $G : \mathbb{R}^m \to \mathbb{R}^n$ and $(dG)_{\phi(a)}$ is surjective for all $a \in G^{-1}(b)$, then in appropriate coordinates $G : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_n)$. So $G^{-1}(0) = \{(0, \dots, 0, x_{n+1}, \dots, x_m)\}$. The $x_{n+1}, \dots, x_m$ define the structure of a manifold of dimension $m-n$ on $G^{-1}(0)$.

More formally: WLOG $G(x_1, \dots, x_m) = (x_1, \dots, x_n)$ and $\phi(b) = 0$. Thus $\phi(F^{-1}(b) \cap U) = G^{-1}(0, \dots, 0) \subset (0, \dots, x_{n+1}, \dots, x_m)$. Write $\phi(a) = (\phi_1(a), \phi_2(a))$ where $\phi_1(a) \in \mathbb{R}^n$ (the first n coordinates) and $\phi_2(a) \in \mathbb{R}^{m-n}$ (the last $m-n$ coordinates). A chart on $F^{-1}(b) \cap U$ is given by $(F^{-1}(b) \cap U, \phi_2)$and ϕ_2 maps $F^{-1}(b) \cap U$ to an open ball in $\mathbb{R}^{m-n}$.

We check that these charts form a C^∞-compatible atlas: If we have another $(\tilde{U}, \tilde{\phi})$, write $\tilde{\phi} = (\tilde{\phi_1}, \tilde{\phi_2})$ and $\tilde{\phi_2} \circ \tilde{\phi_1}^{-1} = \Pi \circ \tilde{\phi} \circ \phi^{-1} \circ i$ where $\Pi : \mathbb{R}^m \to \mathbb{R}^{m-n}$ is projection on the last $m-n$ coordinates, while $i : \mathbb{R}^{m-n} \to \mathbb{R}^n$ is inclusion as the last $m-n$ coordinates. Hence $\tilde{\phi_2} \circ \tilde{\phi_1}^{-1}$ is C^∞, since $\tilde{\phi} \circ \phi^{-1}$ is.

The proof of the following result is similar to the proof of the regular value theorem:

Theorem 4.10 *(Constant rank theorem) If $F : M^m \to N^n$ is smooth and F has constant rank r on a neighbourhood of a for every $a \in F^{-1}(b)$, then $F^{-1}(b)$ is a submanifold of M of dimension $m-r$.*

Example 4.11 $F : \mathbb{R}^m \to \mathbb{R}$, $F(x) =< x, x >$.

$$(dF)_x(\xi) = 2 < \xi, x >$$

So $(dF)_x$ is onto $\mathbb{R}$ unless $x = 0$. Hence any $b \neq 0$ is a regular value of F. The corresponding $F^{-1}(b)$ are manifolds of dimension $m-1$ (they are diffeomorphic to S^{m-1}).

Example 4.12

$$O(n) = \{A \in M_{n\times n} : A^t A = 1\}$$

Define

$$F : M_{n\times n} \to S_n \cong \mathbb{R}^{n+\frac{n(n-1)}{2}}$$

by

$$F(A) = A^t A$$

so

$$(dF)_A(\xi) = \xi^t A + A^t \xi$$

Claim: The identity matrix I is a regular value of F.
Proof: (of Claim) We want to check that for all symmetric matrices C there exists ξ for which $(dF)_A(\xi) = C$. Put $\xi = AC/2$. Then

$$(dF)_A(\xi) = \frac{C^t A^t A}{2} + \frac{A^t A C}{2} = C.$$

Theorem 4.13 *(Sard's Theorem) The set of critical values of a C^∞ map $f : M \to N$ has Lebesgue measure* 0*.*

Definition 4.14 *(Submanifolds) Let M be a manifold of dimension m. The space $N \subset M$ is an embedded submanifold of M of dimension n iff for all $x \in N$ there is a coordinate chart (U, ϕ) around x in M in which $\phi(U \subset N) = \phi(U) \cap \mathbb{R}^n$ where $\mathbb{R}^n$ is identified with $\{(z, 0) \in \mathbb{R}^m : z \in \mathbb{R}^n\}$.*

Definition 4.15 *If a map $i : N \to M$ is an injective immersion, $i(N)$ is called an* immersed submanifold *of M. If i is also a homeomorphism onto $i(N)$, then $i(N)$ is called an* embedded submanifold*.*

This is equivalent to the assertion that for all open $U \subset N$ there is an open $V \subset M$ such that $F(U) = V \cap F(N)$ is open in the relative topology on $F(N)$, or equivalently F^{-1} is continuous using this topology.

Example 4.16 *The figure-eight is an example of an immersion of $\mathbb{R}$ into $\mathbb{R}^2$ which is not injective.*

$$F(t) = (2\cos(g(t) - \pi/2), \sin 2(g(t) - \pi/2))$$

where $\lim_{t\to-\infty} g(t) = 0$ and $\lim_{t\to\infty} g(t) = 2\pi$, while $g(0) = \pi/2$.

Example 4.17 *An injective immersion of $\mathbb{R}$ into $\mathbb{R}^2$ which is not a homeomorphism onto its range. (Homeomorphism $\leftrightarrow$ For all $V \subset M$ there is $U \subset N$ s.t. $i(N) \cap V = i(N \cap U)$).*

Example 4.18 *The skew line: $f : \mathbb{R} \to S^1 \times S^1$*

$$f(t) = (e^{it}, e^{i\alpha t}).$$

If α is irrational then the image of f is dense in $S^1 \times S^1$ so if V is an open neighbourhood of $f(t)$ in $S^1 \times S^1$ then

$$\overline{V \cap f(\mathbb{R})} = V$$

so $V \cap f(\mathbb{R}) \neq f(U)$.

Proposition 4.19 *If $F : M \to N$ is an injective immersion and M is compact then F is an embedding and $F(M)$ is a submanifold.*

Proposition 4.20 *If F is an injective immersion and F is proper (in other words the inverse image of a compact set is compact) then F is an embedding and $F(M)$ is a submanifold.*

Proposition 4.21 *If $F : M \to N$ is an immersion, then each $p \in N$ has a neighbourhood U such that $F|_U$ is an embedding of U in N.*

Definition 4.22 Manifolds with Boundary: *A manifold with boundary is a topological space M with a collection of charts (V_α, ϕ_α) with*

$$\phi_\alpha : V_\alpha \to U_\alpha \subset \mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and every point has a neighbourhood homeomorphic to an open subset of $\mathbb{H}^n$. The boundary of M is the set of points which are mapped to $\partial H^n := \{(x_1, \ldots, x_{n-1}, 0)\}$.

Example 4.23 $\{(x \in \mathbb{R}^2 || x| \leq 1\}$ *is a manifold with boundary. The boundary is* $S^1 = \{x \in \mathbb{R}^2 || x| = 1\}$.

5 Tangent spaces

We have already defined smooth maps $F : M \to N$. Now we define an appropriate domain and range for dF, without reference to charts.

The tangent space T_aM to M at a point $a \in M$ can be exhibited as follows.

- If M is an embedded submanifold of $\mathbb{R}^n$, the inclusion map $U \to \mathbb{R}^n$

$$\begin{array}{ccc} U \subset M & \xrightarrow{\;i\;} & \mathbb{R}^n \\ \downarrow \phi & & \downarrow \\ \phi(U) \subset \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \end{array}$$

  Then $T_aM = \mathrm{Im} d(i \circ \phi^{-1}) \subset \mathbb{R}^n$.

- If $M = F^{-1}(0)$ for $F : \mathbb{R}^n \to \mathbb{R}^m$, then $T_aM = (dF)_a^{-1}(0)$.

We will revisit these two examples.

Example 5.1 *S^n is $F^{-1}(1)$ for $F : \mathbb{R}^n \to \mathbb{R}$ given by $F(x) =< x, x >$. Then $(dF)_a^{-1}(0) = \{\xi \in \mathbb{R}^n | < a, \xi >= 0\}$ (this is the plane orthogonal to a).*

Three definitions of the tangent space
First definition of tangent space:

Definition 5.2 *(Short Curves) A* short curve *γ at a is a smooth map $\gamma : (-\delta, \delta) \to M$ with $\gamma(0) = a$.*

Definition 5.3 *Two short curves γ_1, γ_2 at a are tangent to each other if for a chart (U, ϕ) we have $(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$. We can check that this is independent of the choice of charts.*

Definition 5.4 *A* tangent vector *is an equivalence class of mutually tangent short curves at a. The set of such equivalence classes is denoted S_a.*

Note one may identify $\gamma_1 : (-\delta_1, \delta_1) \to M$ and $\gamma_2 : (-\delta_2, \delta_2) \to M$ even if $\delta_1 \neq \delta_2$.

Remark 5.5 *Any chart (U, ϕ) gives a map $T_\phi : S_a \to \mathbb{R}^n$ defined by*

$$T_\phi(\gamma) = (\phi \circ \gamma)'(0).$$

This allows the tangent space to be identified with $\mathbb{R}^n$.

Second definition of tangent space: Suppose (U, ϕ) is a local chart and $x_i = \pi_i \circ \phi_U$ (projection on the i-th coordinate).

Likewise $y_i = \pi_i \circ \phi_V$ for another compatible coordinate chart (V, ϕ_V).

Define

$$T_a M := \{(x, v) | \sim\}$$

where $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ and $x = (x_1, \ldots, x_n)$. $(x, v) \sim (y, w)$ if

$$w = d(\phi_V \circ \phi_U^{-1})_{\phi_U(a)}(v)$$

Less formally: Tangent vectors are

$$\sum_j v_j \frac{\partial}{\partial x_j}$$

with the equivalence relation that

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}$$

($\frac{\partial}{\partial y_j}$ is a notation for the j-th basis vector in $\mathbb{R}^n$). The vector space structure is as follows:

$$[x, v] + [x, w] = [x, v + w]$$

$$\lambda[x, v] = [x, \lambda v]$$

(for $\lambda \in \mathbb{R}$).

The identification between Definition 1 and Definition 2 is as follows: $[x, v]_a$ is identified with the equivalence class of $\phi_u^{-1} \circ \gamma$ where γ is a curve in $\mathbb{R}^n$ with $\gamma'(0) = v$.

The vector space structure is transferred from $\mathbb{R}^n$ to the space of equivalence classes of curves: $\sum_i a_i \frac{\partial}{\partial x_i}$ corresponds to the curve $[t \mapsto \phi_u^{-1}(t \sum_i a_i e_i)]$. The basis element $\frac{\partial}{\partial x_i}$ is identified with $\gamma_i(t) = \phi_u^{-1}(te_i)$.

Third definition of tangent space:

$$T_a M = \{X : C^\infty(U) \to \mathbb{R} | X(fg) = (Xf)g(a) + f(a)(Xg)$$

Here X is a derivation.

Claim: $T_a M$ is the span of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$.

Lemma 5.6 *If f is smooth in a convex neighbourhood of 0 in $\mathbb{R}^n$ and $f(0) = 0$, then there is $g_i : U \to \mathbb{R}^n$ with*

1. $f(x_1, \ldots, x_n) = \sum_{i=1}^n x_i g_i(x_1, \ldots, x_n)$

2. $g_i(0) = \frac{\partial}{\partial x_i} f(0)$.

Proof: Define $h_x(t) = f(tx)$ for $0 \leq t \leq 1$. Then $f(x) = f(x) - f(0) = \int_0^1 \frac{\partial}{\partial t} h_x(t) dt$

$$= \int_0^1 \sum_i (\frac{\partial}{\partial x_i} f)(tx) w \cdot x_i dt$$

(by the chain rule). So set $g_i(x) = \int_0^1 \frac{\partial}{\partial x_i} f(tx) dt$

Proof of Claim:

For any derivation X

$$X(1) = X(1 \cdot 1) = 1 \cdot X(1) + 1 \cdot X(1)$$

so $X(1) = 0$. So $\forall f$ defined on a convex domain around 0, where $f(0) = 0$, $Xf = X(f - f(0)) = X \sum_i x_i g_i = \sum_i (X x_i) g_i(0) + x_i(0)(X g_i)) = \sum_i (X x_i) \frac{\partial f}{\partial x_i}(0)$ Hence $\partial / \partial x_i$ span the vector space.

Stereographic projection of S^2 on $\mathbb{R}^2$ creates two coordinate systems:

$$u = \frac{x}{1+z}, v = \frac{y}{1+z}$$

and

$$\hat{u} = \frac{x}{1-z}, \hat{v} = \frac{y}{1-z}$$

Then

$$u/\hat{u} = (1-z)/(1+z) = \frac{1-z^2}{(1+z)^2}$$

$$= u^2 + v^2$$

so $\hat{u} = \frac{u}{u^2+v^2}$ and $\hat{v} = \frac{v}{u^2+v^2}$. Likewise

$$\hat{u}/u = (1+z)/(1-z) = \frac{1-z^2}{(1-z)^2}$$

$$= \hat{u}^2 + \hat{v}^2$$

so $u = \frac{\hat{u}}{\hat{u}^2+\hat{v}^2}$ and $v = \frac{\hat{v}}{\hat{u}^2+\hat{v}^2}$ So

$$\frac{\partial}{\partial \hat{u}} = \frac{\partial}{\partial u}\frac{\partial u}{\partial \hat{u}} + \frac{\partial}{\partial v}\frac{\partial v}{\partial \hat{v}}$$

$$= \left(\frac{1}{\hat{u}^2 + \hat{v}^2} - 2\frac{\hat{u}^2}{(\hat{u}^2+\hat{v}^2)^2} \right) \frac{\partial}{\partial u} - \frac{2\hat{u}\hat{v}}{(\hat{u}^2+\hat{v}^2)^2} \frac{\partial}{\partial v}.$$

Example 5.7 *Polar coordinates on* $\mathbb{R}^2$*:*

$$x = r\cos(\theta), y = r\sin(\theta)$$

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}$$

$$= \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y}$$

$$= -r\sin(\theta)\frac{\partial}{\partial x} + r\cos(\theta)\frac{\partial}{\partial y}$$

We can also express the x, y *coordinates in terms of the* r, θ *coordinates:* $r = \sqrt{x^2 + y^2}, \theta = \arctan(y/x)$ *so*

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$$

$$= \frac{x}{r}\frac{\partial}{\partial r} + \frac{1}{(y/x)^2 + 1}\left(\frac{-y}{x^2}\right)\frac{\partial}{\partial \theta}$$

and similarly

$$\frac{\partial}{\partial y} = \frac{y}{r}\frac{\partial}{\partial r} + \frac{1}{(y/x)^2 + 1}\left(\frac{1}{x}\right)\frac{\partial}{\partial \theta}$$

Transformation properties of tangent spaces under $F : M \to N$

1. Curves: $dF[t \mapsto \gamma(t)] = [t \mapsto F \circ \gamma(t)]$

2. Local coordinates: Writing coordinates $(z_1, \ldots, z_n)$ on a chart in N, and $z_j \circ F = F_j$,

$$dF(\frac{\partial}{\partial x_i}) = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_i}\frac{\partial}{\partial z_j}.$$

3. Point derivations: $dF(X)(g) = X(g \circ F)$

Less formally, if we choose coordinates on chart domains in M and N then dF is given by the Jacobian matrix $\frac{\partial F_j}{\partial x_i}$.

Example 5.8 $F:\mathbb{R}^2\to\mathbb{R}^3$ $F(x,y)=(u,v,w)$ *where* $u=xy$, $v=x+3y$, $w=x^2y^2$

$$dF(\frac{\partial}{\partial x}) = \frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}+\frac{\partial w}{\partial x}\frac{\partial}{\partial w}$$
$$= y\frac{\partial}{\partial u}+\frac{\partial}{\partial v}+2x^2y\frac{\partial}{\partial w}$$
$$dF(\frac{\partial}{\partial y}) = \frac{\partial u}{\partial y}\frac{\partial}{\partial u}+\frac{\partial v}{\partial y}\frac{\partial}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial}{\partial w}$$
$$= x\frac{\partial}{\partial u}+3\frac{\partial}{\partial v}+2x^2y\frac{\partial}{\partial w}$$

Of course, a change of coordinates can be viewed as a diffeomorphism.

For example, we can view polar coordinates as a map $F:\mathbb{R}^+\times[0,2\pi]\to\mathbb{R}^2\setminus\{0\}$ $F(r,\theta)=(r\cos\theta, r\sin\theta)=(x,y)$. So $dF(\partial/\partial r)=\frac{\partial x}{\partial r}\frac{\partial}{\partial x}+\frac{\partial y}{\partial r}\frac{\partial}{\partial y}=-r\sin\theta\frac{\partial}{\partial x}+r\cos\theta\frac{\partial}{\partial y}$

Proposition 5.9 $d(F\circ G)_p=(dF)_{G(p)}\circ(dG)_p$ *for*

$$M \xrightarrow{G} N \xrightarrow{F} P$$

Proof: This is immediate in terms of the first definition (since a short curve γ in M is sent to the curve $F\circ G(\gamma)$). In the second definition (using local coordinates), it is a consequence of the chain rule. After all, dF reduces to the Jacobian matrix $d(\Psi\circ f\circ\phi^{-1})$

5.1 The cotangent space

The space of cotangent vectors T_p^*M is the dual space of T_pM. A basis for T_p^*M is given by the differentials dx_i corresponding to the coordinate functions x_i. These give the dual basis to the basis $\{\frac{\partial}{\partial x_i}\}$ for T_pM. Transformation under a change of coordinates:

$$dx_j=\sum_i\frac{\partial x_j}{\partial y_i}dy_i=\sum_i B_{ji}dy_i$$

where the matrix B is given by

$$B_{ji}=\frac{\partial x_j}{\partial y_i}.$$

For any smooth function $f:M\to\mathbb{R}$, $(df)_p:T_pM\to\mathbb{R}$ is a cotangent vector, given by

$$df=\sum_j\frac{\partial f}{\partial x_j}dx_j.$$

Remark 5.10 *We shall see that the matrix B is $(T^{-1})^t$ where $T_{ji} = \frac{\partial y_i}{\partial x_j}$. The matrix T transforms bases of tangent vectors:*

$$\frac{\partial}{\partial x_j} = \sum_i T_{ji} \frac{\partial}{\partial y_i}$$

To prove this, we observe that

$$\sum_i \frac{\partial y_i}{\partial x_j} \frac{\partial x_k}{\partial y_i} = \delta_{jk}$$

(where $\delta_{jk} = 1$ when $j = k$ and 0 otherwise).

$$dx_i = \sum_j \frac{\partial x_i}{\partial y_j} dy_j = \sum_j P_{ij} dy_j$$

and

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j} = \sum_j Q_{ij} \frac{\partial}{\partial y_j}$$

where $P_{ij} = \frac{\partial x_i}{\partial y_j}$ and $Q_{ij} = \frac{\partial y_j}{\partial x_i}$. We check that $\sum_j P_{ij} Q_{kj} = \delta_{ik}$. Hence $P = (Q^{-1})^T$.

Example 5.11 *On S^2, near $(0,0,1)$ we can take coordinates x, y. Define $z = \sqrt{1 - x^2 - y^2}$. Then*

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ &= \frac{1}{z}(-x dx - y dy) \end{aligned}$$

5.2 Transformation properties under maps:

- Tangent vectors push forward:

  $f : M \to N$ gives

  $$(df)_p : T_p M \to T_{f(p)} N$$

  If $(z_1, \ldots, z_n) : V \to \mathbb{R}$ are coordinates on a chart in N, and $(x_1, \ldots, x_m) : U \to \mathbb{R}$ in M,

  $$(df)_p(\frac{\partial}{\partial x_i}) = \sum_j \frac{\partial}{\partial x_i}(z_j \circ f) \frac{\partial}{\partial z_j}.$$

  Alternative notation for $(df)_p$ is $(f_*)_p$.

- Cotangent vectors pull back:

  If $f : M \to N$, and dg is a cotangent vector on N $(dg)_q \in T_q^*N$, then $f^*((dg)_{f(p)}) = d(g \circ f)_p$. In particular $f^*dz_j = d(z_j \circ f) = \sum_i \frac{\partial(z_j \circ f)}{\partial x_i} dx_i$

Example 5.12 $f : \mathbb{R}^2 \to \mathbb{R}^3$

$$f(x, y) = (F_1(x, y), F_2(x, y), F_3(x, y)$$

where $F_1(x, y) = xy$, $F_2(x, y) = x + 3y$, $F_3(x, y) = y^2x^2$

$f^*(dF_1) = d(F_1 \circ f) = d(xy) = ydx + xdy$

$f^*(dF_2) = d(F_2 \circ f) = dx + 3dy$

$f^*(dF_3) = d(F_3 \circ f) = d(y^2x^2) = 2yx^2dy + 2xy^2dx$

6 Vector bundles

A *vector bundle* over a manifold B is a triple $(E.B, \pi)$ where

1. B is a manifold (base space)
2. E is also a manifold (total space)
3. $\pi : E \to B$ is a surjective map (bundle projection) such that for each $b \in B$, $\pi^{-1}(b)$ is endowed with the structure of an n-dimensional vector space such that E is *locally trivial* (i.e. for each $b \in B$ there is a neighbourhood U containing b such that $E|_U := \pi^{-1}(U)$ is isomorphic to $U \times \mathbb{R}^n$

(Bundle isomorphism: A collection of maps

$$\begin{array}{ccc} E_1 & \xrightarrow{\Psi} & E_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ B & \xrightarrow[=]{} & B \end{array}$$

The map Ψ is bijective. The restriction of Ψ to each $\pi^{-1}(b)$ is a linear mapping.
Ψ is called a *bundle chart*.

6.1 Construction of bundles

All bundles may be constructed by imposing an equivalence relation

$$E = \{U \times \mathbb{R}^n | \sim\}$$

where $(a, \xi) \sim (a, g_{VU}(a)\xi)$ if $a \in U \cap V$. Here the $g_{UV} : U \cap V \to GL(n, \mathbb{R})$ satisfy

- $g_{UU} = \mathrm{id}$
- $g_{UV} \cdot g_{VW} = g_{UW}$
- $g_{UV} \cdot g_{VU} = \mathrm{id}$

Example 6.1 *The tangent bundle: Suppose U (resp. V) are coordinate charts with coordinates $\{x_i\}$ (resp. $\{y_j\}$). The transition functions*

$$(g_{UV})_{ij} = \frac{\partial y_i}{\partial x_j}$$

are the transition functions for the tangent bundle TM.

6.2 Sections

A *section* s is a smooth map $s : B \to E$ such that $\pi \circ s = \mathrm{id}$.

A section of the tangent bundle is called a vector field.

One obvious section is the zero section $s(b) = 0 \; \forall b$.

A bundle is trivial if $E = B \times \mathbb{R}^N$

A bundle is trivial iff it has a basis of global nowhere vanishing sections $s_i : B \to \mathbb{R}^n$ so that $\Psi : B \times \mathbb{R}^n \to E$ defined by

$$\Psi(b, (x_1, \ldots, x_n)) = \sum_i x_i s_i(b)$$

is a bundle isomorphism.

In some cases it is impossible to find even *one* nowhere vanishing section of a vector bundle (for example the tangent bundle of S^2). In terms of transition functions, the sections $s_U : U \to \mathbb{R}^n$ satisfying $g_{UV} s_V = s_U$ on $U \cap V$

6.3 Complex vector bundles

Analogous definition: but each fibre $\pi^{-1}(b)$ has the structure of a complex vector space and the bundle charts are subsets of $\mathbb{C}^n$.

7 Differential forms

Let M be a manifold with $\dim M = m$. Recall that $\Omega^r M$ is defined as the smooth sections of $\Lambda^r T^* M$. In coordinates $(x_1, \ldots, x_m)$ on a chart U, an r-form ρ is $\sum_I f_i(x) dx_{i_1} \wedge \ldots dx_{i_r}$ for $i_1 < \ldots < i_r$ If $r > m$, $\Omega^r(M) = \{0\}$. $\Omega^0(M) = C^\infty(M)$. $r = m$: an r-form ρ defines a multilinear function on $T^p M$ for all $p \in M$.

$$(Y_1, \ldots, Y_r) \in T_p M \mapsto \rho(Y_1, \ldots, Y_r) \in \mathbb{R}.$$

This function is linear in each Y_i.

Results about exterior algebras:

If $\rho \in \Omega^r(M)$, $\sigma \in \Omega^s(M)$, then there exists $\rho \wedge \sigma \in \Omega^{r+s}(M)$ satisfying

1. $\rho \wedge \sigma = (-1)^{rs} \sigma \wedge \rho$ (for example if ρ is an odd-degree form then $\rho \wedge \rho = 0$: This is not necessarily the case for even-degree forms)
2. $\wedge$ is bilinear on forms
3. $\wedge$ is associative
4. if f is a smooth function, it may be viewed as a 0-form, so $f \wedge (\sum_I g_I dx_{i_1} \wedge \ldots \wedge dx_{i_r}) = \sum_I (f g_I) dx_{i_1} \wedge \ldots \wedge dx_{i_r}$.

Suppose V is a vector space with a basis $\{v_1, \ldots, v_m\}$ and the dual basis $\{\phi_1, \ldots, \phi_m\}$ for V^*. Then

$$\phi_1 \wedge \ldots \wedge \phi_m = m! \mathrm{Alt}(\phi_1 \otimes \ldots \otimes \phi_m) := \sum_\pi (-1)^\pi \phi_{\pi(1)} \otimes \ldots \otimes \phi_{\pi(m)}.$$

Then $\phi \wedge \ldots \phi_m(v_1, \ldots, v_m) = 1$ (only one permutation contributes). In particular for $V = T_p M$, $V^* = T_p^* M$ $\phi_i = dx_i, v_i = \frac{\partial}{\partial x_i}$.

$$dx_1 \wedge \ldots \wedge dx_n(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) = 1.$$

Assume $1 \le i_1 < \ldots < i_r \le m$, $1 \le j_1 < \ldots < j_r \le m$:

$$dx_{i_1} \wedge \ldots \wedge dx_{i_r}(\frac{\partial}{\partial x_{j_1}}, \ldots, \frac{\partial}{\partial x_{j_r}}) = 0$$

unless $\{i_1, \ldots, i_r\} = \{j_1, \ldots, j_r\}$ in which case the result is 1.

Example 7.1

$$dx_1 \wedge dx_2(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}) = 1$$

$$dx_1 \wedge dx_2(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}) = -1$$

$$dx_1 \wedge dx_2(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}) = 0$$

Example 7.2 *($(n-1)$-form on $S^{n-1} \subset \mathbb{R}^n$)*

$$\omega_{(x_1,\ldots,x_n)} = \sum_{i=1}^{n} (-1)^i x_i dx_0 \wedge \ldots \wedge d\hat{x}_i \wedge \ldots \wedge dx_n$$

where $x_i : \mathbb{R}^n \to \mathbb{R}$ For example, on $S^1 \subset \mathbb{R}^2$ $\omega = xdy - ydx$. It will turn out that ω is the natural volume form on S^{n-1}.

7.1 Exterior differential:

$d : \Omega^r(M) \mapsto \Omega^{r+1}(M)$ is defined by

$$d(a_I dx_{i_1} \wedge \ldots \wedge dx_{i_r}) = \sum_{\ell} (\frac{\partial a_I}{\partial x_\ell} dx_\ell \wedge dx_{i_1} \wedge \ldots dx_{i_r}).$$

Note that

1. $dx_\ell \wedge dx_\ell = 0$ so if ℓ appears among $\{i_1, \ldots, i_r\}$ then the term involving $\frac{\partial a_I}{\partial x_\ell}$ vanishes.
2. $dx_i \wedge dx_j = -dx_j \wedge dx_i$.

The following Lemma follows from properties of exterior algebras.

Lemma 7.3 *$F^*(\omega \wedge \theta) = F^*\omega \wedge F^*\theta$.*

We have

Theorem 7.4

$$dF^* = F^* d.$$

This is proved as follows.
Proof:

(1) True if $\omega = g$ is a C^∞ function, for $d(F^*g) = d(g \circ F) = F^*(dg)$

(2) True if $\omega = dg$ for $F^*\omega = F^*dg = d(g \circ F)$ and $d(F^*\omega) = 0$, also $F^*d(dg) = 0$.

(3) If $F^*d\theta = d(F^*\theta)$ for $\theta \in \Omega^r(M)$, and $F^*(d\omega) = d(F^*\omega)$, then $d(\theta \wedge \omega) = (d\theta) \wedge \omega + (-1)^r \theta \wedge d\omega$ so $F^*d(\theta \wedge \omega) = F^*(d\theta) \wedge F^*\omega + (-1)^r (F^*\theta) \wedge (F^*d\omega)$ (by the Lemma) $= d(F^*\theta) \wedge F^*\omega + (-1)^r F^*\theta \wedge d(F^*\omega)$ (by hypothesis) $= d(F^*\theta) \wedge F^*\omega = d(F^*(\theta \wedge \omega))$ (by Lemma)

(4) By induction on r: use the fact that all r-forms are of the form

$$a_I(x) dx_{i_1} \wedge \ldots \wedge dx_{i_r}.$$

Also (1) gives the result for a_I while (2) and (3) combine to give the result for $dx_{i_1} \wedge \ldots \wedge dx_{i_r}$.

Example 7.5 $F : \mathbb{R}^2 \to \mathbb{R}^3$ $F(x,y) = (F_1(x,y), F_2(x,y), F_3(x,y))$ *where*

$$F_1(x,y) = x^2 \sin(y)$$

$$F_2(x,y) = y^3 e^{2x}$$

$$F_3(x,y) = xy$$

Then

$$F^*dz_1 = dF_1 = \frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy = 2x \sin y dx + x^2 \cos y dy$$

$$F^*dz_2 = \frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy = 2y^3 e^{2x} dx + 3y^2 e^{2x} dy$$

$$F^*(dz_1 \wedge dz_2) = F^*dz_1 \wedge F^*dz_2 = \big((2x \sin y)(3y^2 e^{2x}) - (2y^3 e^{2x})(x^2 \cos y)\big) dx \wedge dy$$

Very important result:

Theorem 7.6 *On $\mathbb{R}^n$, for any smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$ we have*

$$F^*(dy_1 \wedge \ldots \wedge dy_n) = \det(dF)(dx_1 \wedge \ldots dx_n).$$

This follows by the determinant theorem.

The importance of this will appear when we reach integration on manifolds: if g is a smooth function on $\mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$, and $y = F(x)$, then

$$\int_{F(A)} g(y) dy = \int_A g(F(x)) \frac{dF}{dx} dx.$$

This is best summarized by viewing $g(y)dy$ as a 1-form so

$$F^*(g(y)dy) = g(F(x))\frac{dF}{dx}dx.$$

Proof: (of Theorem)

Write $F = (F_1, \ldots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$. Then

$$F^*(dy_1 \wedge \ldots \wedge dy_n) = dF_1 \wedge \ldots \wedge dF_n = \sum_{i_1,\ldots,i_n} \frac{\partial F_1}{\partial x_{i_1}} \ldots \frac{\partial F_n}{\partial x_{i_n}} dx_{i_1} \wedge \ldots \wedge dx_{i_n}$$

$$= \sum_{\pi} \frac{\partial F_1}{\partial x_{\pi(1)}} \ldots \frac{\partial F_n}{\partial x_{\pi(n)}} dx_{\pi(1)} \wedge \ldots \wedge dx_{\pi(n)}$$

$$= \sum_{\pi} (-1)^{\pi} \frac{\partial F_1}{\partial x_{\pi(1)}} \ldots \frac{\partial F_n}{\partial x_{\pi(n)}} dx_1 \wedge \ldots \wedge dx_n$$

$$= \det(dF) dx_1 \wedge \ldots \wedge dx_n$$

since dF is the $n \times n$ matrix whose (i,j) entry is $\frac{\partial F_i}{\partial x_j}$.

7.2 A useful differential form

On $S^n \subset \mathbb{R}^{n+1}$ a nowhere zero differential form is given by

$$\omega_{x_0,\ldots,x_n} = \sum_j (-1)^j x_j dx_1 \wedge \ldots \wedge \hat{dx_j} \wedge \ldots \wedge dx_n.$$

For example, on S^1 we recover $x_0 dx_1 - x_1 dx_0 = r^2 d\theta$ in polar coordinates.

7.3 The exterior differential

Definition 7.7 *The exterior differential $d : \Omega^r(M) \to \Omega^{r+1}(M)$ is defined by*

$$d(\sum_I a_I dx_{i_1} \wedge \ldots \wedge dx_{i_r}) = \sum_{\ell} \sum_I \frac{\partial a_I}{\partial x_\ell} dx_\ell \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r}.$$

Remark 7.8

1. *$dx_\ell \wedge dx_\ell = 0$ so if ℓ appears among $\{i_1, \ldots, i_r\}$ then the term involving $\frac{\partial a_I}{\partial x_\ell}$ vanishes.*

2. *$dx_i \wedge dx_j = -dx_j \wedge dx_i$.*

Properties:

1. d is linear
2. if f is a smooth function, df is a 1-form
3. $\rho \in \Omega^r(M), \sigma \in \Omega^s(M) \Rightarrow d(\rho \wedge \sigma) = (d\rho) \wedge \sigma + (-1)^r \rho \wedge d\sigma$
4. $d^2 = 0$

Proof: (of (3)) $d(a_I dx^I \wedge b_J dx^J) = d(a_I b_J) \wedge dx^I \wedge dx^J$

$$= (da_I b_J + a_I db_J)(dx^I \wedge dx^J)$$

(by Leibniz)

$$= (d\rho) \wedge \sigma + \rho \wedge d\sigma$$

Proof: (of (4))

$$d^2(a_I dx^I) = d(\sum_\ell \frac{\partial a_I}{\partial x^\ell} dx^\ell \wedge dx^I$$

$$= \sum_{k,\ell} \frac{\partial^2 a_I}{\partial x_k \partial x_\ell} dx_k \wedge dx_\ell \wedge dx_I$$

But

$$\frac{\partial^2}{\partial x_k \partial x_\ell} = \frac{\partial^2}{\partial x_\ell \partial x_k}$$

while

$$dx_k \wedge dx_\ell = -dx_\ell \wedge dx_k$$

The following is proved in Guillemin-Pollack:

Proposition 7.9 *d is the unique operator with properties (1)-(4).*

Remark 7.10 *If* $\dim(M) = m$ *and* $\omega \in \Omega^m(M)$ *then* $d\omega = 0$.

Definition 7.11 *A form α is* closed *if $d\alpha = 0$.*

Definition 7.12 *A form α is* exact *if $\alpha = d\beta$ for some form β.*

An important consequence of the fact that $d^2 = 0$ is that exact forms are closed. We can define *de Rham cohomology* as follows:

Definition 7.13 *The k-th de Rham cohomology group of a manifold M is*

$$\frac{\{\alpha \in \Omega^k M | d\alpha = 0\}}{\{\alpha \in \Omega^k M | \alpha = d\beta\}}$$

Example 7.14 *Forms that are closed but not exact:*

- *do not exist on $\mathbb{R}^n$ (Poincaré lemma)*
- *On $S^{n-1} \subset \mathbb{R}^n \setminus \{0\}$ the form $\sum_{i=1}^n (-1)^i x_i dx_1 \wedge \ldots \wedge \hat{dx_i} \wedge \ldots \wedge dx_n$ is closed but not exact. This will follow from Stokes' theorem.*

7.4 Consequences in dimension 3:

Vector fields	**Forms**
F_0 function	$\Omega_0 = F_0$ function
$F_1 = (v_1, v_2, v_3)$ vector field	$\omega_1 = v_1 dx_1 + v_2 dx_2 + v_3 dx_3$ 1-form
$F_2 = (f_1, f_2, f_3$ vector field	$\omega_2 = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$ 2-form
F_3 function	$\omega_3 = F_3 dx_1 \wedge dx_2 \wedge dx_3$
$F_1 = \nabla F_0$	$\omega_1 = d\omega_0$
$F_2 = \nabla \times F_1 = \mathrm{curl} F_1$	$\omega_2 = d\omega_1$
$F_3 = \nabla \cdot F_2 = \mathrm{div} F_2$	$\omega_3 = d\omega_2$

Here $\mathrm{curl} F_1 = (h_1, h_2, h_3)$ where

$$h_1 = \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}$$

$$h_2 = \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}$$

$$h_1 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$$

in other words

$$h_1\hat{e_1} + h_2\hat{e_2} + h_3\hat{e_3} = \det\begin{pmatrix} \hat{e_1} & \hat{e_2} & \hat{e_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{pmatrix}$$

Also

$$divF_2 = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

Hence $d \circ d = 0$ translates to

$$\nabla \times (\nabla F_0) = 0$$

and

$$\nabla \cdot (\nabla \times F_1) = 0$$

8 Transversality

Let V_1 and V_2 be vector subspaces of a vector space V. Then V_1 and V_2 are transversal if $V = V_1 + V_2$ as subspaces of V. If V_1 and V_2 are transversal, it follows that

$$dimV = dimV_1 + dimV_2 - dim(V_1 \cap V_2).$$

Submanifolds N_1 and N_2 of a manifold M are said to be transverse if

$$T_x N_1 + T_x N_2 = T_x M$$

for all $x \in N_1 \cap N_2$.

Whether or not N_1 and N_2 are transversal in M depends on M as well as on N_1 and N_2. For example the x and y axes are transversal in $\mathbb{R}^2$ but not in $\mathbb{R}^3$. If the sum of the dimensions of N_1 and N_2 is less than the dimension of M, then N_1 and N_2 can only intersect transversally if their intersection is empty.

Proposition 8.1 *If N_1 and N_2 are transverse submanifolds of M then $N_1 \cap N_2$ is a submanifold of M of dimension $dimN_1 + dimN_2 - dimM$, or* $\operatorname{codim}(N_1 \cap N_2) = \operatorname{codim}(N_1) + \operatorname{codim}(N_2)$.

Two maps $g_1 : N_1 \to M$ and $g_2 : N_2 \to M$ are transverse if $g_{1*}(T_{x_1}N_1) + g_{2*}(T_{x_2}N_2) = T_y M$ for all x_1, x_2, y with $g_1(x_1) = g_2(x_2) = y$.

Let $\Phi : M \to N$ be a smooth map, and $S \subset N$ an embedded submanifold. Then Φ is transversal to S iff for all $p \in \Phi^{-1}(S)$, $\Phi_* T_p M$ and $T_{\Phi(p)}S$ span $T_{\Phi(p)}N$.

Theorem 8.2 *If $f : M \to N$ is transverse to a submanifold L of codimension k (i.e. dimN - dim L = k) and $f^{-1}(L)$ is not empty, then $f^{-1}(L)$ is a codimension k submanifold of M.*

Proof: Let $f(p) = q \in L$. In some neighbourhood V of q, $L \cap V = \mathbb{R}^{n-k} \cap V'$ (where $\mathbb{R}^{n-k} = \{(x_1, \ldots, x_{n-k}, 0, \ldots, 0)\}$). Define $\pi : \mathbb{R}^n \to \mathbb{R}^k$ to be the projection on the last k coordinates $\pi(x_1, \ldots, x_n) = (x_{n-k+1}, \ldots, x_n)$. The transversality condition means that $0 \in \mathbb{R}^k$ is a regular value of

$$U \xrightarrow{f} V \cong V' \xrightarrow{\pi} \mathbb{R}^k$$

Hence for an open neighbourhood U in M, $f^{-1}(L) \cap U$ is a codimension k submanifold of U (by the Regular Value Theorem). It follows that $f^{-1}(L)$ is a codimension k submanifold of M. If Φ is transversal to S, then $\Phi^{-1}(S)$ is an embedded submanifold of M whose codimension is $dim(N) - dim(S)$. □

Sard's Theorem: Let $f : M \to N$ be a smooth map. Then the set of critical values of f has measure zero in N.

Proof: See Guillemin-Pollack, Appendix 1

Theorem 8.3 *(Whitney embedding theorem) Every k-dimensional manifold admits an embedding in $\mathbb{R}^{2k+1}$.*

Proof (from Guillemin and Pollack chap. 1.8):

We first give an argument that shows that if there is an injective immersion from X to $\mathbb{R}^M$ then there is an injective immersion from X to $\mathbb{R}^{2k+1}$. The hypothesis that there is an injective immersion from X to $\mathbb{R}^M$ (for some M) will be proved in the section 'Partitions of Unity'.

If $f : X \to RR^M$ is an injective immersion with $M > 2k+1$, then there is $a \in \mathbb{R}^M$ such that $\pi \circ f$ is an injective immersion. (Here π is the projection from $\mathbb{R}^M$ to H, where $H = \{b \in \mathbb{R}^M : b \perp a\} \cong a\} \cong \mathbb{R}^{M-1}$. So we have an injective immersion into $\mathbb{R}^{M-1}$. Here define $h : X \times X \times \mathbb{R} \to \mathbb{R}^M$ by

$$h(x, y, t) = t(f(x) - f(y))$$

and $g : TX \to \mathbb{R}^M$ by

$$g(x, v) = df_x(v).$$

Since $dim(M) > 2k+1$, Sard's theorem tells us that there is $a \in \mathbb{R}^M$ which is not in the image of g or the image of h. (Sard's theorem tells us that the regular values of g and h in the image of a smooth map F from an n- manifold to a vector space V of dimension higher than n is dense. When the dimension of V is higher than n, the regular values are the complement of the image of F.)

Let π be the projection of $\mathbb{R}^M$ on H.

Lemma 8.4 *$\pi \circ f : X \to H$ is injective.*

Proof: Suppose not. Then $\pi \circ f(x) = \pi \circ f(y)$ implies $f(x) - f(y) = ta$ for some t. If $x \neq y$ then $t \neq 0$ since f is injective. But then $h(x, y, t^{-1}) = a$ contradicting the choice of a. Likewise

Lemma 8.5 *$\pi \circ f : X \to H$ is an immersion.*

Proof: Suppose v is a nonzero vector in T_xX with $d(\pi \circ f)_x(v) = 0$. π is linear, so $d(\pi \circ f) = \pi \circ df$. So $\pi \circ df_x(v) = 0$ implies $df_x(v) = ta$ for some t. Since f is an immersion, $t \neq 0$. Hence $g(x, t^{-1}a) = a$, contradicting the choice of a. □

(Note that in fact Whitney eventually proved that it was possible to embed a k-dimensional manifold in $\mathbb{R}^{2k}$.)

9 The Lie derivative

Let α be an r-form and let β be an s-form on M. Let X be a vector field on M. Define the *interior product* $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ by

$$(i_X\alpha)(Y_1, \ldots, Y_{k-1}) = \alpha(X, Y_1, \ldots, Y_{k-1})$$

(we insert X as the first argument of α). This is also called the *contraction* of α by X. Then

$$i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge i_X\beta$$

Remark 9.1 *If X is a vector field and $f : M \to \mathbb{R}$ is a smooth function then*

$$i_{fX}\alpha)(p) = f(p)(i_X\alpha)(p)$$

(in other words the interior product is $C^\infty(M)$ -linear in X).

Also, We have already shown that

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta.$$

Define the Lie derivative by

$$L_X = di_X + i_Xd$$

which sends r-forms to r-forms. A straightforward calculation shows that

$$L_X(\alpha \wedge \beta) = (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta).$$

A derivation *is a linear map L from $\Omega^r M$ to $\Omega^{r+t}M$ (here t is independent of r) satisfying*

$$L(\alpha \wedge \beta) = (L\alpha) \wedge \beta + \alpha(L\beta).$$

An antiderivation *is a linear map A from $\Omega^r M$ to $\Omega^{r+t}M$ (here t is independent of r) satisfying*

$$A(\alpha \wedge \beta) = (A\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha(A\beta).$$

For example d and i_X are antiderivations. The above shows that L_X is a derivation. The formula $L_X = di_X + i_Xd$ together with the fact that i_X is $C^\infty(M)$-linear in X show that L_X is not $C^\infty(M)$-linear in X. Instead

$$(L_{fX}\alpha)(p) = f(p)(L_X\alpha)(p) + (df) \wedge (i_X\alpha)(p)$$

Definition 9.2 *(Lie bracket of vector fields) In terms of evaluation of vector fields on functions,*

$$[X, Y]f = X(Yf) - Y(Xf)$$

In a coordinate system, all vector fields are of the form

$$X = \sum_i a_i(x)\frac{\partial}{\partial x_i}, Y = \sum_i b_i(x)\frac{\partial}{\partial x_i}$$

so

$$\begin{aligned} X(Yf) &= \sum_i a_i(x)\frac{\partial}{\partial x_i}\left(\sum_j b_j\frac{\partial f}{\partial x_j}\right) \\ &= \sum_{i,j} a_i\left(\frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} + b_j\frac{\partial^2 f}{\partial x_i \partial x_j}\right). \end{aligned}$$

Similarly

$$Y(Xf) = \sum_{i,j} b_j\left(\frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} + a_i\frac{\partial^2 f}{\partial x_i \partial x_j}\right).$$

Hence

$$[X,Y]f = \sum_{i,j}\left(a_i\frac{\partial b_j}{\partial x_i} - b_i\frac{\partial a_j}{\partial x_i}\right)\frac{\partial f}{\partial x_j}.$$

It follows immediately that

Proposition 9.3

$$[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$$

The following is obvious:

Lemma 9.4 $[X,Y] = -[Y,X]$

The following can be proved by calculation with the above formula for the Lie bracket of vector fields:

Lemma 9.5 *(Jacobi identity)*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

There is a formula for the exterior derivative in terms of the Lie bracket (see Boothby Chap. V, (8.4)). Here is the special case of the exterior derivative on 1-forms.

Proposition 9.6 *If X and Y are vector fields on M and α is a 1-form then*

$$(d\alpha)(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]).$$

Proof: WLOG $\alpha = fdg$ for smooth functions f and g. So $d\alpha = df \wedge dg$.

$$d\alpha(X, Y) = df(X)dg(Y) - dg(X)df(Y)$$
$$= (Xf)(Yg) - (Xg))(Yf)$$

while

$$X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) = X(fdg(Y)) - Y(fdg(X)) - fdg([X, Y])$$
$$= X(f(Yg)) - Y(f(Xg)) - f([X, Y]g)$$

By the Leibniz rule, this becomes

$$= (Xf)(Yg) + f(X(Yg)) - (Yf))(Xg) - f(Y(Xg)) - f(X(Yg) - Y(Xg))$$

Four terms cancel, and we obtain

$$(Xf)(Yg) - (Xg))(Yf)$$

as claimed.

The more general formula is (if α is an r-form)

$$d\alpha(X_1, \ldots, X_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i-1} X_i \alpha(X_1, \ldots, \hat{X}_i, \ldots, X_{r+1})$$
$$+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{r+1}.)$$

10 Flows

Let X be a vector field on a manifold M.

Definition 10.1 *An* integral curve *of X is a smooth map $F : (a, b) \to M$ for which*

$$\frac{dF}{dt}(t_0) = X_{F(t_0)}$$

Let W be an open set in $\mathbb{R} \times M$ satisfying $\exists \alpha(p) < 0 < \beta(p)$ such that

$$W \cap (\mathbb{R} \times \{p\}) = \{(t, p) : \alpha(p) < t < \beta(p)\}$$

Definition 10.2 *A* local flow *on M is a smooth map $\theta : W \to M$ such that (introducing the definition $\theta_t(p) := \theta(t, p)$)*

1. $\theta_0(p) = p \forall p \in M$

2. If $(s, p) \in W$ $\alpha(\theta_s(p)) = \alpha(p) - s$ and $\beta(\theta_s(p)) = \beta(p) - s$, and for t such that $\alpha(p) - s < t < \beta(p) - s$, $\theta_{t+s}(p)$ is defined and

$$\theta_t(\theta_s(p)) = \theta_{t+s}(p).$$

   In particular θ_t is a local diffeomorphism (the inverse is θ_{-t}).

Equivalently, we have a collection of open neighbourhoods V_α covering M and maps $\theta^\alpha : (-\epsilon_\alpha, \epsilon_\alpha) \times V_\alpha \to M$ such that

$$\theta^\alpha_t(p) := \theta^\alpha(t, p)$$

such that

1. θ^α and θ^β agree on the intersection of their domains

2. $\theta^\alpha(0, p) = p$

3. $\theta^\alpha_{t+s} = \theta^\alpha_t \circ \theta^\alpha_s$ where both sides are defined

Theorem 10.3 *Given a vector field X on M, there exists $\theta : (-\delta, \delta) \times V \to M$ for which*

$$\frac{d}{dt}\theta(t, p) = X_{\theta(t,p)}$$

and $\theta(0, p) = p$ for all $p \in V$, in other words $\{\theta(t, p)\}$ is an integral curve of X through p. Any two such θ are equal on the intersection of their domains.

Proof: Follows from existence and uniqueness of solutions for ordinary differential equations.

Theorem 10.4 *1. Let $U \subset \mathbb{R}^n$, $(f_1, \ldots, f_n) : (-\epsilon, \epsilon) \times U \to \mathbb{R}^n$ smooth. Then there exists an open subset V of U such that for any $(a_1, \ldots, a_n) \in V$ there exist $(x_1, \ldots, x_n) : (-\epsilon, \epsilon) \to U$ satisfying*

(a) $\frac{dx_i}{dt} = f_i(t, (x_1(t), \ldots, x_n(t)))$

(b) $x_i(0) = a_i$ for $i = 1, \ldots, n$.

The functions x_i are uniquely determined.

2. Write $x_i(t, (a_1, \ldots, a_n))$. Then x_i is smooth in t and $(a_1, \ldots, a_n)$.

Proof: (of existence of integral curves:) The vector field X is written in local coordinates $(y_1, \ldots, y_n)$ as $(f_1, \ldots, f_n)$. Then

$$X = \sum_i f_i \frac{\partial}{\partial y_i}$$

If $\theta = (\theta^1, \ldots, \theta^n) : W \to M$ then solving

$$\frac{\partial \theta_t}{\partial t} = X$$

is equivalent to solving

$$\frac{\partial \theta^i}{\partial t} = f_i$$

The existence of solutions of this follows from existence and uniqueness of ODE's.

Lemma 10.5 *If $I(p) = \{(\alpha(p), \beta(p))\}$ where the flow $\theta(t, p)$ is defined for $t \in (\alpha(p), \beta(p))$, then we assume the domain W is maximal (in other words that $|\alpha(p)|$ and $|\beta(p)|$ are maximal for all p).*

Lemma 10.6 *If $\beta(p) < \infty$ and t_n is an increasing sequence converging to $\beta(p)$, then $\{\theta(t_n, p)\}$ cannot lie in any compact set.*

Proof: Let $K \subset M$ be a compact set. By the existence theorem, for all $q \in M$ there exists $\delta > 0$ and a neighbourhood V of q such that θ is defined on $I_\delta \times V$. A finite number of these cover K. Let δ_0 be the minimum δ for these neighbourhoods. So for any $q \in K$ $\theta(t, q)$ is defined for $|t| < \delta_0$. If $\{(\theta(t_n, p)\} \subset K$, and N so large that $\beta(p) - t_N < \delta_0/3$, then $\theta(t, \theta(t_N, p))$ is defined only for $t < \beta(p) - t_N < \delta_0/3$. But $\theta(t_N, p) \in K$ so $\theta(t, \theta(t_N, p))$ is well defined for $|t| < \delta_0$. $\square$

Definition 10.7 *A vector field is* complete *if $\theta(t,p)$ is defined for all $p \in M$ and all $t \in \mathbb{R}$.*

Corollary 10.8 *(of Lemma) If M is compact then any vector field on M is complete.*

Example 10.9 *If M is a compact manifold and X is a vector field on M consider $M' = M \smallsetminus \{p\}$. The flow generated by $X|_{M'}$ is not complete. Take $y \in M$ for which $\theta(t_0, y) = p$, so $y = \theta(-t_0, p)$. Then on M', $\theta(t,y)$ is only defined for $t < t_0$.*

Theorem 10.10 *1. Lie derivative on forms:*

$$L_X\omega = \frac{d}{dt}|_{t=0}\theta_t^*\omega = \lim_{t\to 0}\frac{1}{t}(\theta_t^*\omega - \omega)$$

2. Special case of above: $L_X f = Xf$

3. Lie derivative on vector fields:

$$L_X Y = -\frac{d}{dt}|_{t=0}(\theta_t)_* Y = \lim_{t\to 0}\frac{1}{t}(Y - (\theta_t)_* Y_{\theta_{-t}})$$

Proposition 10.11 *If X and Y are vector fields, then*

$$L_X Y = [X, Y]$$

Proof:

$$(L_X Y)\circ f = \lim_{t\to 0}\frac{1}{t}\left(Y(0)(f) - (\theta_t)_* Y_{\theta_{-t}(0)}(f)\right)$$

$$= \lim_{t\to 0}\frac{1}{t}\left(Y(0)(f) - Y_{\theta_{-t}(0)}(f\circ\theta_t)\right)$$

But $\theta_t(p) = p + tX(p) + O(t^2)$ so $f\circ\theta_t(p) = f(p) + tdf(X(p)) + O(t^2)$ and we get

$$\lim_{t\to 0}\frac{1}{t}\left(Y(0)f - Y_{\theta_{-t}(0)}(f + tdf(X)(p) + O(t^2)\right)$$

$$= \lim_{t\to 0}\frac{1}{t}\left(Y_0 f - Y_{\theta_{-t}(0)} f\right) = X(Yf) - Y(Xf) = [X,Y]f$$

Proposition 10.12 *(Cartan's theorem) On differential forms,*

$$L_X = di_X + i_X d.$$

Proof: Since L_X and di_X and i_Xd are derivations, it suffices to prove the identity on the differential forms f (0-form) and df (exact 1-form) since all differential forms are given locally as wedge products of forms of this type.

1. $L_X f = i_X(df)$: The left hand side is

$$\frac{d}{dt}\theta_t^* f = \frac{d}{dt} f \circ \theta_t = df(X)$$

   This completes the proof.

2. $L_X df = di_X df$: The left hand side is

$$\frac{d}{dt}|_0 \theta_t^* df = d/dt|_0 d(f \circ \theta_t) = d(i_X df).$$

   This completes the proof.

Proposition 10.13 *If X is a smooth vector field with $X(p) \neq 0$ then there are coordinates $x : U \to \mathbb{R}^n$ on a neighbourhood U of p for which $X = \partial/\partial x_1$ on U. The flow in these coordinates is*

$$\theta_t(x_1, \ldots, x_n) = (x_1 + t, x_2, \ldots, x_n).$$

Proof: Start with an open set $U \subset M$. Assume we have coordinates $(y_1, \ldots, y_n) : U \to \mathbb{R}^n$ with $y(p) = 0$. Define $\theta_t(q)$ the flow of X starting at $q \in M$. Assume $X(p) = \frac{\partial}{\partial y_1}(p)$. We shall define new coordinates using the flow of X. In a neighbourhood of 0 there is a unique integral curve through each point $y = (0, a_2, \ldots, a_n)$. If q lies on the integral curve through this point, use $a_2, \ldots, a_n$ as the last $n-1$ coordinates and the time it takes the curve to get to q as the first coordinate. Define new coordinates $x : V \to \mathbb{R}^n$ on a neighbourhood V of p by $x^{-1}(a_1, \ldots, a_n) = \theta_{a_1} \circ y^{-1}(0, a_2, \ldots, a_n)$, in other words $a_j = x_j \circ \theta_{a_1} \circ y^{-1}(0, a_2, \ldots, a_n)$, and

$$\frac{\partial}{\partial x_1}\left(y_i \circ x^{-1}(0, \ldots, 0)\right) = \lim_{t \to 0} \frac{1}{t}\left(y_i \circ \theta_t \circ y^{-1}(0, \ldots, 0) - y_i \circ \theta_0 \circ y^{-1}(0, \ldots, 0)\right)$$

$= (dy_i)(X(p)) = \delta_{i1}$ since $X(p) = \frac{\partial}{\partial y_1}$.

$$\frac{\partial}{\partial x_j}(y_i \circ x^{-1})(0, \ldots, 0) = \lim_{t \to 0} \frac{1}{t}\left(y_i \circ \theta_0 \circ y^{-1}(0, \ldots, t, \ldots, 0) - y_i \circ \theta_0 \circ y^{-1}(0, \ldots, 0)\right)$$

(where the t in the first expression is the j-th coordinate) $= \delta_{ij}$ since θ_0 is the identity so $y \circ \theta_0 \circ y^{-1}$ is also the identity. Hence $d(y \circ x^{-1})_{(0,\ldots,0)}$ is nonsingular (the Jacobian determinant is nonzero), which implies $y \circ x^{-1}$ is a local diffeomorphism (by the inverse function theorem). Hence x are local coordinates (since y are).

Proposition 10.14 *In the preceding situation,* $\frac{\partial}{\partial x_1} = X$.

Proof: For $f : U \to \mathbb{R}$,

$$\left(\frac{\partial}{\partial x_1} f\right)_{x^{-1}(a_1,\ldots,a_n)} = \lim_{h\to 0} \frac{1}{h}\left(f(x^{-1}(a_1+h,\ldots,a_n)) - f(x^{-1}(a_1,\ldots,a_n))\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left[f\left(\theta_{a_1+h}\circ y^{-1}(0,a_2,\ldots,a_n)\right) - f\left(\theta_{a_1}\circ y^{-1}(0,a_2,\ldots,a_n)\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[f\left(\theta_h(p')\right) - f(p')\right]$$

where $p' = \theta_{a_1}\circ y^{-1}(0,a_2,\ldots,a_n) = x^{-1}(a_1,\ldots,a_n))$. This then equals $(Xf)(p')$ since $\theta_t(q)$ is the flow of X starting at q.

Lemma 10.15 *Two flows θ_t and ψ_s commute if and only if the corresponding vector fields commute.*

Proof: See Boothby IV.7.12

Definition 10.16 *Suppose* $\dim M = n+k$. *A* distribution *on M is an assignment of an n-dimensional subspace Δ_p of T_pM at each $p \in M$. Suppose in a neighbourhood U of each $p \in M$ there are n linearly independent vector fields $X_1,\ldots,X_n$ which form a basis of Δ_q for each $q \in U$. Then $\{X_i\}$ are called a* local basis *of Δ.*

Example 10.17 $M = \mathbb{R}^{n+k}$, *Δ spanned by $\frac{\partial}{\partial x_i}$, $i = 1,\ldots,n$.*

Definition 10.18 *A distribution is* integrable *if each point has a coordinate neighbourhood $(x_1,\ldots,x_m)$ for which a local basis for Δ is given by $\{\frac{\partial}{\partial x_i}, i = 1,\ldots,m\}$.*

Definition 10.19 *A distribution is* involutive *if there exists a local basis in a neighbourhood of each point such that*

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$$

in other words $[X_i, X_j]_p$ lies in the plane Δ_p for all $p \in M$.

Theorem 10.20 (Frobenius) *A distribution is integrable iff it is involutive.*

Note that 'if' is clear since

$$[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0.$$

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11 Smooth functions and partitions of unity

11.1 Smooth functions

Example 11.1 *The function* $\theta : \mathbb{R} \to \mathbb{R}$ *given by*

$$\theta(t) = 0, t \leq 0$$

and

$$\theta(t) = e^{-1/t}, t > 0$$

is smooth and all its derivatives are 0 *at* $t = 0$*. In particular it is not represented by its Taylor series at* 0*.*

The open cube $C(r)$ is defined as follows.

Definition 11.2

$$C(r) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | |x_i| < r \forall i\}$$

The closure of $C(r)$ *is denoted* $\overline{C(r)}$*.*

Lemma 11.3 *There exists a smooth function* $h : \mathbb{R}^n \to \mathbb{R}$ *with*

1. $0 \leq h(x) \leq 1$
2. $h(x) = 1,\ x \in \overline{C(1)}$
3. $h(x) = 0,\ x \notin C(2)$

Remark 11.4 *The function* h *is called the* bump function*.*

Proof: Define

$$\phi(x) = \frac{\theta(x)}{\theta(x) + \theta(1-x)}.$$

Then $\phi(x) = 1$ for $x > 1$ and $\phi(x) = 0$ for $x \leq 0$. Define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(x) = \phi(x+2)\phi(2-x).$$

Then $\psi(x) = 1, |x| \leq 1$ while $\psi(x) = 0, |x| \geq 2$. Thus define $h(x_1, \ldots, x_n) = \psi(x_1) \ldots \psi(x_n)$.

Definition 11.5 *The* support *of a smooth function* $f : M \to \mathbb{R}$ *is the closure of the set of points* $x \in M$ *where* $f(x) \neq 0$.

Consequences of existence of the function h:

Proposition 11.6 *Let* M *be a smooth manifold and let* (U, ϕ) *be a chart in an atlas for* M*. There exists a smooth function* $f : M \to \mathbb{R}$ *with* $f(M) \subset [0, 1]$ *and* $\mathrm{Supp}(f) \subset U$*, and* $f(x) = 1$ *on a neighbourhood of* $p \in U$*.*

Proof: For a point $p \in U$ choose a cubical neighbourhood $B \subset \mathbb{R}^n$ around $\phi(p)$, say

$$\{x : |\phi(p_i) - x_i| < \epsilon\}.$$

Define $\alpha : B \to C(2)$ by

$$\alpha(x) = \frac{2}{\epsilon}(x - \phi(p))$$

and define

$$f(x) = \{h \circ \alpha \circ \phi(x), x \in U \cap \phi^{-1}(B)\}$$

and 0 otherwise. Then $h : \mathbb{R}^n \to \mathbb{R}$, $h(x) = 1$ if $|x_i| \leq 1$ for all i, and $h(x) = 0$ if $|x_i| \geq 2$ for some i.

Definition 11.7 *A partition of unity subordinate to an open cover* $\{U_\alpha\}$ *of* M *is a collection of smooth functions* $f_\gamma : M \to \mathbb{R}$ *such that*

1. *For all* γ, $\mathrm{Supp}(f_\gamma) \subset U_\alpha$ *for some* α *(here* $\mathrm{Supp} f_\gamma$ *is the closure of the subset where* $f_\gamma(x) \neq 0$*)*

2. $0 \leq f_\gamma \leq 1$ *on* M

3. $\forall x \in M$ *there is an open neighbourhood* V_x *of* x *s.t. there exist only finitely many* f_γ *s.t.* $\mathrm{Supp} f_\gamma \cap V_x \neq \emptyset$ *are nonzero at any points on* V_x

4. $\sum_\gamma f_\gamma(x) = 1$ *(this sum is finite because of (3))*

We shall prove existence of a partition of unity. We require some facts from general topology.

Lemma 11.8 *Manifolds are* regular *(in other words if* $C \subset X$ *is a closed subset,* $C \neq X$ *and* $x \in X \setminus C$ *then these can be separated by disjoint open subsets)*

Let M be a Hausdorff space.

Definition 11.9 *M is* paracompact *if*

1. *M is regular*
2. *every open cover admits a locally finite refinement*

Definition 11.10 *An open cover $\{\mathcal{V}\}$ is a refinement of the open cover $\{\mathcal{U}\}$ if there exists $\imath : \mathcal{I}_V \to \mathcal{I}_U$ (where $\mathcal{I}_V$ is the indexing set of $\{\mathcal{V}\}$ and similarly for $\{\mathcal{U}\}$) such that $V_\beta \subset I_{\imath(\beta)}$.*

Theorem 11.11 *Manifolds are paracompact.*

Proof: There exist compact subsets $K_1 \subset K_2 \subset \dots$ of M such that $K_r \subset \mathrm{Int}(K_{r+1}$ and $M = \cup_r \mathrm{Int}(K_r)$. Let $\{W_i\}$ be a countable base of the topology with each $\bar{W}_i$ compact. $K_1 = \bar{W}_1, \dots, K_r \subset \cup_{i=1}^{\ell} W_i$ (let ℓ be the smallest for which this is true) and $K_{r+1} = \cup_{i=1}^{\ell} \bar{W}_i$. Let $\{U_\alpha\}$ be an open cover: to get a locally finite refinement, choose finitely many $V_i = U_{\alpha_i}$ covering K_1. Extend this by $\{U_{\alpha_i}\}_{i=\ell_1+1}^{\ell_2}$ to give an open cover of K_2. M is Hausdorff so K_1 is closed, and $V_i = U_{\alpha_i} \setminus K_1$ is open, $\ell_1 + 1 \leq \ell_2$ and $\{V_i\}_{i=\ell_1+1}^{\ell_2}$ is an open cover of K_2.

Note that K_1 does not meet V_i for $i > \ell_1$.

By induction we get $\{V_i\}$ such that K_r meets only finitely many elements of $\mathcal{V}$ $\forall r \geq 1$.

For any $x \in M$, $x \in \mathrm{Int}(K_r)$ for some r, there exists a neighbourhood meeting only finitely many elements of $\mathcal{V}$.

Definition 11.12 *A precise refinement of an open cover $\{U_\alpha\}$ is a locally finite refinement indexed by the same set with $\bar{V}_\alpha \subset U_\alpha$.*

Proposition 11.13 *If M is a paracompact manifold and $\{U_\alpha\}$ is an open cover of M, then this cover has a precise refinement.*

Proof: There exists a refinement $\{W_k\}$ with $j : \mathcal{K} \to \mathcal{A}$ such that $\bar{W}_k \subset U_{j(k)}$ (since M is regular). Passing to a locally finite refinement of $\mathcal{W}$ gives a locally finite refinement $\mathcal{V}'$ of $\mathcal{U}$ with $\imath : \mathcal{B} \to \mathcal{A}$ with $\bar{V}'_\beta \subset U_{\imath(\beta)}$. The $\bar{V}'_\beta$ are a locally finite family of closed subsets of M. For all $\alpha \in \mathcal{A}$, define $\beta_\alpha := \imath^{-1}(\alpha)$.

$$V_\alpha = \cup_{\beta \in \beta_\alpha} V'_\beta$$

Because $\mathcal{V}'$ is locally finite, $\bar{V}_\alpha = \cup_{\beta \in \mathcal{B}_\alpha} \bar{V}'{}_\beta \subset U_\alpha$.

Definition 11.14 *M is normal if whenever A and B are disjoint closed subsets of M, there is an open set U containing A and disjoint from B with $\bar{U} \cap B = \emptyset$.*

Lemma 11.15 *Paracompact spaces are normal.*

Proposition 11.16 (Urysohn's lemma) *Suppose M is normal and A and B are closed subsets of M. There exists a smooth function $f : M \to [0,1]$ such that $f|_A = 1$ and $f|_B = 0$.*

Theorem 11.17 *Suppose K is compact and $K \subset U$ for an open set U. Then there exists a smooth function $f : \mathbb{R}^n \to [0,1]$ with $f|_K = 1$ and f supported in U.*

Use Lemma 11.3 to show that

Lemma 11.18 *If $A = (a_1, b_1) \times \ldots \times (a_n, b_n) \subset \mathbb{R}^n$ then there is a smooth function $g_A : \mathbb{R}^n \to [0,1)$ such that $g_A > 0$ on A and $g_A|_{\{\mathbb{R}^n \setminus A\}} = 0$.*

Proof: (of Theorem) Let $K \subset \mathbb{R}^n$ be compact and $U \subset \mathbb{R}^n$ an open neighbourhood of K. For each $x \in K$, let A_x be an open bounded neighbourhood of x of the form

$$A_x = (a_{1,x}, b_{1,x}) \times \ldots \times (a_{n,x}, b_{n,x})$$

with $\bar{A+x} \subset U$, $x \in A_x$. By Lemma 11.18, there is a smooth function $g_x : \mathbb{R}^n \to [0,1)$ with $g_x(y) > 0$ for $y \in A_x$ and $g_x(y) = 0$ for $y \neq A_x$. Since K is compact, it is covered by finitely many $A_{x_1}, \ldots, A_{x_q}$. Define $G = g_{A_{x_1}} + \ldots + g_{A_{x_q}} : \mathbb{R}^n \to \mathbb{R}$. Then G is smooth on $\mathbb{R}^n$, $G(x) > 0$ if $x \in K$ and $\text{supp}(G) = \bar{A}_{x_1} \cup \ldots \cup \bar{A}_{x_q} \subset U$. Since K is compact, there exists $\delta > 0$ such that $G(x) \geq \delta$ for $x \in K$. Define our bump function ℓ so it is 0 for $t \leq 0$ and 1 for $t \geq \delta$. Define $f = \ell \circ G : \mathbb{R}^n \to [0,1]$. Then

1. f is smooth
2. $\text{supp}(f) \subset U$
3. $f|_K = 1$

Theorem 11.19 *There is a partition of unity subordinate to any open cover $\mathcal{U}$.*

Proof: If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then a partition of unity subordinate to $\mathcal{V}$ induces one subordinate to $\mathcal{U}$.

$$\imath : \mathcal{B} \to \mathcal{U}$$

$V_\beta \subset U_{\imath(\beta)}$ $\{\mu_\beta\}$ subordinate to $\mathcal{U}$

$$\lambda_\alpha = \sum_{\beta \in \imath^{-1}(\alpha)} \mu_\beta.$$

So since manifolds are locally compact, WLOG each U_α has compact closure in M.

A precise refinement has the property that $\bar{V_\alpha} \subset U_\alpha$ is a compact subset. We may use Urysohn's lemma to give f. We choose a precise refinement $\mathcal{V}$ with $\mathcal{W}$ a precise refinement of $\mathcal{V}$.

$\{\gamma_\alpha\}_{\alpha \in U}$ satisfy $\gamma_\alpha|_{W_\alpha} = 1$ and $\text{supp}(\gamma_\alpha) \subset \bar{V_\alpha} \subset U_\alpha$.

$\{\text{supp}\gamma_\alpha\}$ is locally finite. $\gamma := \sum_\alpha \gamma_\alpha$ is smooth and > 0. Define $v_\alpha = \gamma_\alpha/\gamma$, which is smooth. The v_α are a partition of unity.

Applications of partitions of unity:

The primary application is integration on manifolds. Let us begin with integration of a function on $\mathbb{R}^n$. Assume $\{U_\alpha\}$ is an open cover of $\mathbb{R}^n$ and consider a partition of unity $\{f_\alpha\}$ subordinate to this open cover. Let g be a smooth function on $\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} g = \int_{\mathbb{R}^n} (\sum_\alpha f_\alpha) g = \sum_\alpha \int_{U_\alpha} (f_\alpha g).$$

Application to Whitney embedding theorem:

Proposition 11.20 *Let X be a compact manifold. Then there is an injective immersion from X into $\mathbb{R}^M$ for some M.*

Proof: Construct a covering of X by charts (U_α, ϕ_α), and take a partition of unity $\{f_\alpha\}$ subordinate to the covering $\{U_\alpha\}$. Since X is compact, WLOG we may assume the number of U_α is a finite number M. Then define $F : X \to \mathbb{R}^M$ by

$$F(x) = (f_1(x)\phi_1(x), \ldots, f_M(x)\phi_M(x))$$

12 Orientations and volume forms

Definition 12.1 *Let V be a vector space of dimension n. The top exterior power $\Lambda^n V^*$ has dimension 1 so $\Lambda^n V^* \setminus \{0\}$ has two components. An* orientation *of V is the choice of one of these. Equivalently, an orientation on V is the choice of an ordered basis $[e_1, \ldots, e_n]$ of V with $[e_1, \ldots, e_n]$ declared equivalent to $[f_1, \ldots, f_n]$ if the linear map $B : V \to V$ defined by $Be_i = f_i$ has determinant > 0.*

Example 12.2 *If (e_1, e_2) is the usual ordered basis for $\mathbb{R}^2$, then the two orientations of $\mathbb{R}^2$ are $[e_1, e_2]$ and $[e_2, e_1]$.*

Example 12.3 *If e_1, e_2 are the first two basis vectors for $\mathbb{R}^3$, then the two possible orientations of $\mathbb{R}^3$ are $[e_1, e_2, e_1 \times e_2]$ and $[e_1, e_2, e_2 \times e_1]$ (where $\times$ denotes the cross product).*

An invertible linear map $F : V \to V$ preserves orientation if $\det(F) > 0$.

12.1 Orientation of a manifold

Definition 12.4 *A manifold is* orientable *if we have a covering by charts (U_α, ϕ_α) for which, for any two charts (U, ϕ) and (V, ψ) with coordinate functions $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, we have*

$$detd(\psi \circ \phi^{-1}) > 0.$$

Proposition 12.5 *A manifold M of dimension n is orientable iff it has a nowhere vanishing n-form (call it ω).*

Proof: Suppose such a form ω exists. Then in local coordinates $(x_1, \ldots, x_n)$,

$$\omega = f(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) dx_1 \wedge \ldots \wedge dx_n,$$

and in different local coordinates $y_1, \ldots, y_n$,

$$\omega = f(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}) dy_1 \wedge \ldots \wedge dy_n.$$

Then denoting $w_y := [\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}]$ and $w_x := [\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$ (these are local sections of $\Lambda^n TM$) we have that $\det(\frac{\partial y_j}{\partial x_i}) > 0$ which is the definition of orientability. Suppose on the other hand we know that M is orientable. Suppose $\{U_\alpha, f_\alpha\}$ is a partition of unity subordinate to an open cover $\{U_\alpha\}$ of M. Suppose that (x_α) are coordinates on U_α. We can choose the open sets U_α so that

$\Lambda^n \frac{\partial}{\partial x_\alpha}$ is nonzero on each U_α. Then there is a nowhere vanishing n-form $\omega_\alpha := dx_1 \wedge \ldots \wedge dx_n$ supported on U_α. This lets us make a nowhere vanishing n-form ω on M,

$$\omega := \sum_\alpha f_\alpha \omega_\alpha.$$

Proposition 12.6 *Suppose $M \neq \mathbb{R}^{n+1}$ is an n-dimensional submanifold, given by an embedding $\psi : M \to \mathbb{R}^{n+1}$, and M has a nowhere vanishing normal vector field N (in other words $\forall p \in M\ \exists N(p) \in T_p\mathbb{R}^{n+1}$ smoothly varying as p varies in M, with $N(p) \perp T_pM\ \forall p \in M$) with respect to the Euclidean metric on $\mathbb{R}^{n+1}$). Then M is orientable.*

Proof: We will construct a nowhere vanishing n-form ω. Take

$$\omega_p(X_1, \ldots, X_n) = dx_1 \wedge \ldots \wedge dx_{n+1}(N(p), X_1, \ldots, X_n)$$

Suppose there is p for which $\omega_p = 0$. as an element of $\Lambda^n T_p^* M$. In fact the formula we have given defines $\omega_p(v_1, \ldots, v_n)$ for $v_i \in T_p\mathbb{R}^{n+1}$. But all vectors $v_j \in T_p\mathbb{R}^{n+1}$ can be given as $v_j = \xi_j + a_j N(p)$ for some $\xi_j \in T_pM$, $a_j \in \mathbb{R}$ and $\omega_p(v_1, \ldots, v_n) = \omega_p(\xi_1, \ldots, \xi_n) + \sum_{i=1}^n a_j \omega_p(\xi_1, \ldots, N(p), \ldots, \xi_n)$ (where $N(p)$ is in the j-th place) + terms where at least two arguments are $N(p)$. The terms $\omega_p(\xi_1, \ldots, N(p), \ldots, \xi_n)$ are equal to 0 since $\xi_j \in T_pM$ (because we assumed ω_p vanished on T_pM). Likewise the terms with two or more arguments of ω_p given by $N(p)$ are $= 0$. Write $N(p) = (N_1(p), \ldots, N_{n+1}(p))$. Then $\omega_p(v_1, \ldots, v_n) = 0\ \forall v_j \in T_p\mathbb{R}^{n+1}$. But

$$\omega_p = \sum_{j=1}^n (-1)^j N_j(p) dx_1 \wedge \ldots \wedge \hat{dx_j} \wedge \ldots \wedge dx_{n+1}$$

as a tensor on $\mathbb{R}^{n+1}$ and at least one of the coordinates $N_j(p)$ is nonzero. This is a contradiction. Hence ω_p cannot be zero as a tensor on $\mathbb{R}^{n+1}$, so our assumption that it is zero on T_pM must be false.

Example 12.7 *The volume element on S^2 is given by the restriction to S^2 of the 2-form ω on $\mathbb{R}^3$ given by*

$$\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2$$

Substituting spherical coordinates $x_1 = \sin\theta\cos\phi$, $x_2 = \sin\theta\sin\phi$, $x_3 = \cos\theta$ one recovers

$$\omega = \sin\theta d\theta \wedge d\phi.$$

Definition 12.8 *A vector bundle is orientable if the transition functions g_{UV} can be chosen to satisfy*
$det g_{UV}(y) > 0\ \forall y \in U \cap V,\ \forall U, V.$

For example, the tangent bundle is orientable iff $det d(\psi \circ \phi^{-1}) > 0$ for all chart maps ϕ, ψ. This is the usual definition of the manifold M being orientable.

12.2 Antipodal map on S^n

Let ω be the volume form on S^n, and $i : S^n \to \mathbb{R}^{n+1}$ the inclusion map. $\omega = i^*\Omega$ where Ω is the form

$$\Omega = \sum_{j=1}^{n+1} (-1)^{j-1} x_j dx_1 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots \wedge dx_{n+1}$$

on $\mathbb{R}^{n+1}$. Notice that if $A : S^n \to S^n$ is defined by

$$A(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_{n+1})$$

then $i \circ A = \bar{A} \circ i$ where $\bar{A} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is defined by $\bar{A}(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_{n+1})$. So $A^* i^* \Omega = i^* \bar{A}^* \Omega$. For $F = (F_1, \ldots, F_{n+1}) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and $x_i : \mathbb{R}^{n+1} \to \mathbb{R}$ a coordinate function, we have

$$F^* x_i = x_i \circ F = F_i$$

and

$$F^* dx_i = d(x_i \circ F) = dF_i$$

Applying this to $F = \bar{A} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ we get $\bar{A}^* x_j = -x_j$, $\bar{A}^* dx_j = -dx_j$, and

$$\bar{A}^*\Omega = \sum_{j=1}^{n+1} (-1)^{j-1} (-x_j)(-dx_1) \wedge \ldots \wedge (-dx_{j-1}) \wedge (-dx_{j+1}) \wedge \ldots \wedge (-dx_{n+1}) = (-1)^{n+1}\Omega$$

Thus

$$A^*\omega = i^* \bar{A}^* \Omega = (-1)^{n+1} i^* \Omega = (-1)^{n+1} \omega.$$

So if n is odd there is a nowhere vanishing n-form on $\mathbb{R}P^n$ coming from the volume form ω on S^n, for which $A^*\omega = \omega$. If n is even, we have seen (using a partition of unity) that if a manifold is orientable then it has a nowhere vanishing n-form. So if $\mathbb{R}P^n$ were orientable, there would be a nowhere vanishing n-form on $\mathbb{R}P^n$ and $A^* q^* \omega = q^* \omega$. But any volume form $\hat{\omega}$ on S^n is of the form $\hat{\omega} = f(x)\omega_0$ where ω is the standard volume form on S^n and $f : S^n \to \mathbb{R} \setminus \{0\}$. Hence since $A^*\omega = -\omega$, $A^*(f\omega) = -(f \circ A)\omega$ so if $A^*(f\omega) = f\omega$ (in other words, if $f\omega$ were invariant under the antipodal map) then $f \circ A(x) = -f(x)$, which is impossible. It follows that $\mathbb{R}P^n$ is not orientable if n is even.

Remark 12.9 *The product of two orientable manifolds is orientable.*

Proposition 12.10 *For any manifold M with charts $U \subset \mathbb{C}^n$ and chart transformations $\psi \circ \phi^{-1}$ given by holomorphic maps $f_i : \mathbb{C}^n \to \mathbb{C}^n$, M is orientable.*

Proof: If $n = 1$, $\psi \circ \phi^{-1} = f_1 + i f_2$ where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$. Then

$$d(\psi \circ \phi^{-1}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

The Cauchy-Riemann equations tell us that

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2}$$

$$\frac{\partial f_1}{\partial x_2} = -\frac{\partial f_2}{\partial x_1}$$

This tells us that

$$\det d(\psi \circ \phi^{-1}) = |\frac{\partial f_1}{\partial x_1}|^2 + |\frac{\partial f_2}{\partial x_2}|^2.$$

More generally if a linear map $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ comes from a linear map $F : \mathbb{C}^n \to \mathbb{C}^n$ (i.e. an nxn matrix of complex numbers) this means $F_{\mathbb{R}}$ splits into blocks of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

and these can be reorganized into blocks of the form

$$\begin{pmatrix} F_{\mathbb{C}} & 0 \\ 0 & \bar{F_{\mathbb{C}}} \end{pmatrix}$$

so $\det F_{\mathbb{R}} = |\det F_{\mathbb{C}}|^2 > 0$. We apply this to $F = d(\phi \circ \psi^{-1})$.

Example 12.11 *$\mathbb{C}P^n$ is orientable (because $\phi_j \circ \phi_i^{-1}$ are of the form*

$$(w_1, \ldots, w_n) \mapsto (w_1/w_j, \ldots, 1/w_j, \ldots, \hat{w_j}, \ldots, w_n/w_j)$$

(where $1/w_j$ is in the i-th position and $\hat{w_j}$ is in the j-th position), and these are holomorphic functions of $(w_1, \ldots, w_n)$.

Example 12.12 (Möbius band) *$M = \{[0, 2\pi + \epsilon] \times (-1, +1)\}/\sim$ where $(x, \lambda) \sim (2\pi + x, -\lambda$ for $x \in [0, \epsilon]$. M contains $M' = (0, 2\pi) \times (-1, 1)$ which is orientable. If M had an orientation it would restrict on M' to one of the two standard orientations of M. But the map $(x, \lambda) \mapsto (2\pi + x, -\lambda)$ is orientation reversing, so M cannot have an orientation.*

Example 12.13 *The Klein bottle K is nonorientable since it contains the Möbius band.*

13 Integration on manifolds

Recall that integration of functions over domains U in $\mathbb{R}^n$ is not invariant under diffeomorphism. According to the Change of Variables Theorem in one variable,

$$\int_{f(a)}^{f(b)} h(y)dy = \int_a^b h(f(x))|\frac{df}{dx}|dx.$$

In n variables we have

$$\int_{f(U)} hdy_1 \dots dy_n = \int_U h \circ f|\text{det}df|dx_1 \dots dx_n.$$

But if we write

$$\omega = dy_1 \wedge \dots \wedge dy_n$$

$$f^*\omega = h \circ f(\text{det}df)dx_1 \wedge \dots \wedge dx_n$$

so

$$\int_{f(U)} \omega = \int_U f^*\omega$$

if we define

$$\int hdx_1 \wedge \dots \wedge dx_n = \int_{\mathbb{R}^n} hdx_1 \dots dx_n$$

for any integrable h on $\mathbb{R}^n$.

Let M be an oriented manifold with an oriented atlas with charts (U_α, ϕ_α).

Definition 13.1 *If ω has compact support in U_α and with $\left((\phi_\alpha)^{-1}\right)^* \omega|_{\phi_\alpha(U_\alpha)} = A(x)dx_1 \wedge \dots \wedge dx_n$, then $\int_M \omega := \int_{\phi_\alpha(U_\alpha)} A(x)dx_1 \dots dx_n$. In particular, if ω has compact support in $U \subset \mathbb{R}^m$, then $\omega = A(x)dx_1 \wedge \dots \wedge dx_n$. Or $\int_{\mathbb{R}^n} \omega := \int_U A(x)dx_1 \dots dx_n$.*

Definition 13.2 *Suppose ω is an arbitrary smooth form on M. Let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, then*

$$\int_M \omega := \sum_\alpha \int_M f_\alpha \omega$$

Note that $f_\alpha \omega$ is supported in U_α.

Theorem 13.3 *If $f : M \to N$ is orientation preserving, then $\int_N \omega = \int_M f^*\omega$. If instead f is orientation reversing then $\int_N \omega = -\int_M f^*\omega$.*

Proof:

Case 1: First consider ω supported in $\mathbb{R}^n$: If $\omega(y) = h(y)dy_1 \wedge \ldots \wedge dy_n$ then $f^*\omega(x) = h(f(x))det(df)dx_1 \wedge \ldots \wedge dx_n$. If V is a coordinate chart in $\mathbb{R}^n$ (with coordinates x_i) and $f(V)$ is a coordinate chart with coordinates y_i, then $\int_V f^*\omega = \int_V h(f(x))det(df)dx_1 \ldots dx_n$. So $\int_V f^*\omega = \int_{f(V)} h(y)dy_1 \ldots dy_n$. The two right hand sides are equal by the change of variables theorem.

Case 2: More generally if $(\phi^{-1})^*\omega$ is supported in $\phi(U) \subset \mathbb{R}^n$, then $\int_{f(U)} \omega = \int_{\phi(f(U))} (\phi^{-1})^*\omega$

$$= \int_{\psi(U)} g^*(\phi^{-1})^*\omega$$

(by case 1)

$$= \int_{\psi(U)} (\psi^{-1})^* f^* \phi^*)(\phi^{-1})^*\omega$$

$$= \int_{\psi(U)} (\psi^{-1})^* f^*\omega = \int_U f^*\omega.$$

In particular, taking $f = \text{id}$ but ϕ, ψ arbitrary chart maps compatible with the orientation, we see

$$\int_{\psi(U)} (\psi^{-1})^*\omega = \int_{\phi(U)} (\phi^{-1})^*\omega,$$

so we have

Proposition 13.4 *The integral is well defined independent of the choice of charts.*

Lemma 13.5 *If ω is supported in some chart U then the first definition agrees with the second definition.*

Proof:

$$\sum_\alpha \int_M f_\alpha \omega = \int_{\phi(U)} \sum_\alpha (f_\alpha \circ \phi^{-1})(\phi^{-1})^*\omega) = \int_{\phi(U)} (\phi^{-1})^*\omega$$

The left hand side is Definition 13.2, while the right hand side is Definition 13.1.

Lemma 13.6 *The definition of the integral is independent of the choice of partition of unity.*

Proof: If $\{f_\alpha\}$, $\{g_\beta\}$ are two different partitions of unity then

$$\int_M f_\alpha \omega = \sum_\beta \int_M g_\beta f_\alpha \omega$$

$$\int_M g_\beta \omega = \sum_\alpha \int_M f_\alpha g_\beta \omega$$

so $\sum_\alpha f_\alpha \omega = \sum_\beta g_\beta \omega$.

13.1 Stokes' Theorem

Theorem 13.7 [Stokes] *With the above orientation convention* $\int_M d\omega = \int_{\partial M} \omega$ *if* ω *is an* $(n-1)$*-form on* M*.*

Proof: First assume ω has compact support in U, and $\phi(U)$ is open in $\mathbb{R}^k$ (in other words U contains no boundary points of M). Then $\int_{\partial M} \omega = 0$. We write $\omega = \sum_i f_i dx_1 \wedge \ldots \wedge \hat{dx_i} \wedge \ldots \wedge dx_n$. So

$$d\omega = \sum_i (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n$$

and

$$\int_M d\omega = \sum_i (-1)^{i-1} \int \frac{\partial f_i}{\partial x_i} dx_1 \wedge \wedge dx_n$$

(by definition)

$$= \sum_i (-1)^{i-1} \int dx_1 \wedge \ldots \hat{dx_i} \ldots dx_{n-1} [\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i} dx_i]$$

(by Fubini's theorem)

$$= \sum_i (-1)^{i-1} \int dx_1 \ldots \hat{dx_i} \wedge \ldots dx_{n-1} (f_i(\infty) - f_i(-\infty))$$

(by fundamental theorem of calculus) $= 0$ (as ω is compactly supported).

Now if $U \subset H^n$ (the upper half space), we find instead that

$$\int \frac{df_n}{dx_n} dx_1 \ldots dx_n = (-1)^n f_n(x_1, \ldots, x_{n-1}, 0) dx_1 \ldots dx_{n-1}$$

But this is $\int_{\partial M} \omega$ where we recall that the orientation of ∂M is equal to $[\partial_1, \ldots, \partial_{n-1}]$ where the orientation of M is $[-\partial_n, \partial_1, \ldots, \partial_{n-1}]$. This is then $(-1)^n$ times the orientation $[\partial_1, \ldots, \partial_n]$.

Consequences of Stokes' Theorem in dimension 3: Recall the identification between 1-forms and vector fields (similarly between 2-forms and vector fields).

If γ is a dimension 1 line,

$$\int_\gamma (\nabla g) \cdot \frac{d\gamma}{dt} dt = g(b) - g(a)$$

(fundamental theorem of calculus applied to line integral)

If S_2 is a 2-manifold with boundary in $\mathbb{R}^3$,

$$\int_{S_2} (\nabla \times V_1) \dot{u} A = \int_{\partial S_2} \vec{V_1}$$

(classical Stokes' Theorem)

If B_3 is a 3-manifold with boundary in $\mathbb{R}^3$,

$$\int_{B_3} \nabla \cdot \vec{V_1} dx_1 dx_2 dx_3 = \int_{\partial B_3} \vec{V_2} \cdot \hat{u} A$$

(Gauss's theorem, divergence theorem)

Consequences of Stokes' theorem for vector calculus:

1. Fundamental theorem of calculus (Stokes in dimension 1):

$$\int_a^b f'(x)dx = f(b) - f(a)$$

   This follows from

$$\int_{[a,b]} df = \int_{\partial[a,b]} f = f(b) - f(a).$$

2. Green's theorem (Stokes for $n = 2$ in $\mathbb{R}^2$):

$$\int_{\partial\Omega} (f_1(x,y)dx + f_2(x,y)dy) = \int_{\Omega} (\partial f_2/\partial x - \partial f_1/\partial y))\, dxdy$$

   if Ω is a region in $\mathbb{R}^2$ with boundary $\partial\Omega$ and f_1, f_2 are smooth functions.

   This follows from $\omega = f_1(x,y)dx + f_2(x,y)dy$ and $d\omega = (\partial f_2/\partial x - \partial f_1/\partial y)\, dx \wedge dy$.

3. Gauss's theorem (divergence theorem) (Stokes for $n = 3$): Let M be a 3-manifold with boundary in $\mathbb{R}^3$. The vector field $\vec{F} = (F_1, F_2, F_3)$ on M correponds to a 2-form

$$\omega = F_1(\vec{y})dy_2 \wedge dy_3 + F_2(\vec{y})dy_3 \wedge dy_1 + F_3(\vec{y})dy_1 \wedge dy_2.$$

   So

$$\nabla \cdot \vec{F} dy_1 \wedge dy_2 \wedge dy_3 = d\omega$$

   and

$$(\vec{F} \cdot \vec{u})\mathcal{A} = \omega$$

   where $\mathcal{A}$ is the area form on M and $\hat{u}$ is the unit normal vector to M in $\mathbb{R}^3$.

4. Classical Stokes' theorem: Suppose $\vec{F}$ is a vector field on a 2-manifold Σ embedded in $\mathbb{R}^3$ with boundary $\partial\Sigma$, with ω the corresponding 1-form $\omega = F_1 dx_1 + F_2 dx_2 + F_3 dx_3$. Then

$$\int_{\Sigma} (\vec{\nabla} \times \vec{F}) \cdot \vec{u}\mathcal{A} = \int_{\partial\Sigma} (F_1 dx_1 + F_2 dx_2 + F_3 dx_3).$$

   This translates to

$$\int_{\Sigma} d\omega = \int_{\partial\Sigma} \omega.$$

13.2 Line integral

If $\omega = \sum_i f_i(x_1, x_2, x_3) dx_i$ is a 1-form on $\mathbb{R}^3$ and $\gamma : I \to \mathbb{R}^3$ (where $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ then

$$\int_{\gamma(I)} \omega = \int_I \gamma^* \omega = \int_0^1 \sum_i f_i(\gamma(t)) \frac{d\gamma_i(t)}{dt} dt$$
$$= \int \vec{f} \cdot \frac{d\vec{\gamma}}{dt} dt.$$

The line integral is

$$\int_I \nabla g \cdot \frac{d\vec{\gamma}}{dt} dt = g(1) - g(0).$$

Proposition 13.8 *The line integral over a closed path γ bounding a surface S in $\mathbb{R}^3$ is equal to 0 if ω is defined everywhere on S and $d\omega = 0$.*

Proof: By Stokes, $\int_{\partial S} \omega = \int_S d\omega = 0$.

Remark 13.9 *If $d\omega = 0$ but ω is not defined everywhere, then the conclusion of Proposition 13.8 will not hold: for example*

$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$

on $\mathbb{R}^2 \setminus \{0\}$ but $\int_{S^1} \omega = 2\pi$.

Proposition 13.10 *If ω has the property that $\int_\gamma \omega = 0$ for all closed curves γ, then one can define $\int_p^q \omega$ as $\int_0^1 c^* \omega$ for a curve $c : [0,1] \to \mathbb{R}^3$ with $c(0) = p, c(1) = q$. (This depends only on the endpoints p and q, not on the choice of c.) Any two such curves can be glued to form a closed curve $\gamma = c_1 \cup (-c_2)$ so $\int_0^1 c_1^* \omega = \int_0^1 c_2 * \omega$ since $\int_\gamma \omega = 0$.*

Proposition 13.11 *If $\int_\gamma \omega = 0$ for all closed curves γ on $\mathbb{R}^3$ then $\omega = df$ for some smooth function f.*

Proof: Define $f(x) = \int_p^x \omega$ for any path γ from p to x. (f is well defined by the previous Proposition.) To compute $\partial f / \partial x_1$, take an open neighbourhood of x and

$$\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1} \int_{p_1, x_2, x_3}^{x_1, x_2, x_3} \omega_1(t, x_2, x_3) dt = \omega_1(x_1, x_2, x_3).$$

Similarly to compute $\partial f / \partial x_i$, pick a path where only one coordinate x_j varies at any time and the coordinate x_i changes only in the last segment of the path. We choose a path which is piecewise linear and is obtained by concatenating the line segments v_1, v_2, v_3 where

1. v_1 is the line segment from (p_1, p_2, p_3) to (p_1, p_2, x_3),
2. v_2 is the line segment from (p_1, p_2, x_3) to (p_1, x_2, x_3)
3. v_3 is the line segment from (p_1, x_2, x_3) to (x_1, x_2, x_3).

Remark 13.12 *These propositions generalize to connected manifolds M other than $\mathbb{R}^3$ but we require that $\int_\gamma \omega = 0$ for all closed curves $\gamma \subset M$.*

Proposition 13.13 *If a manifold M is simply connected then every closed 1-form on M is exact.*

Proof: If M is simply connected, then every closed curve $\gamma : S^1 \to M$ extends to a smooth map $\sigma : D^2 \to M$ (where $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$). Then $\int_\gamma \omega = \int_{D^2} d\omega = 0$, so $\int_\gamma \omega = 0$ for every closed curve γ. This is the hypothesis of the previous proposition so $\omega = df$ for some smooth function $f : M \to \mathbb{R}$.

14 Mayer-Vietoris Sequence

If U and V are open subsets of a manifold M, the Mayer-Vietoris sequence is as follows. (We refer to the 'Homological Algebra' notes for the result that a short exact sequence of chain complexes gives rise to a long exact sequence of the corresponding cohomology groups.)

$$0 \longrightarrow \Omega^*(U \cup V) \xrightarrow{f} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{g} \Omega^*(U \cap V) \longrightarrow 0$$

where if $i_U : U \to U \cup V$ and $i_V : V \to U \cup V$, $j_U : U \cap V \to U$, $j_V : U \cap V \to V$ are the canonical inclusions,

$$f = (i_U^*, i_V^*)$$

and

$$g = j_U^* - j_V^*.$$

Clearly f is injective since a differential form α on $U \cup V$ restricting to 0 on both U and V is simply $\alpha = 0$.

To see that g is surjective, we observe that a differential form $\alpha_{U\cap V}$ on $U \cap V$ can written as $\alpha_{U\cap V} = j_U^*\alpha_U - j_V^*\alpha_V$ for suitable forms α_U on U and α_V on V. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$ of $U \cup V$. Then we have $\alpha_{U\cap V} = \rho_U \alpha_{U\cap V} - (-\rho_V)\alpha_{U\cap V}$ on $U \cap V$. Now $\rho_U \alpha_{U\cap V}$ can be extended by zero to give a differential form β_U on U. Likewise $\rho_V \alpha_{U\cap V}$ can be extended by zero to give a differential form β_V on V. Now we have $\alpha_{U\cap V} = j_U^*\beta_U - j_V^*\beta_V$.

Finally we must check that $\mathrm{Im}(f) = \mathrm{Ker}(g)$. This is true because $\mathrm{Ker}(g)$ consists of forms (α_U, α_V) on U (resp. V) with $j_U^*\alpha_U = j_V^*\alpha_V$. This means that α_U and α_V agree on $U \cap V$, so there is a form β on $U \cup V$ with $\alpha_U = i_U^*\beta$ and $\alpha_V = i_V^*\beta$).

This completes the proof that the above sequence is exact.

Then according to the course notes section on 'Homological Algebra' Definition 9.1.1 and Theorem 9.1.12, there is a corresponding long exact sequence of de Rham cohomology groups

$$\longrightarrow H^j(U \cup V) \xrightarrow{f} H^j(U) \oplus H^*(V) \xrightarrow{g} H^j(U \cap V) \xrightarrow{\delta} H^{j+1}(U \cup V) \longrightarrow$$

where δ is the connecting homomorphism. The connecting homomorphism is defined as follows. If $\alpha \in \Omega^j(U \cap V)$ satisfies $d\alpha = 0$, then there are $\alpha_U \in \Omega^j U$ and $\alpha_V \in \Omega^j V$ with $\alpha = j_U^*\alpha_U - j_V^*\alpha_V$. We find that $j_U^* d\alpha_U = j_V^* d\alpha_V$ so there is $\beta \in H^{j+1}(U \cup V)$ with $i_U^*\beta = d\alpha_U$ and $i_V^*\beta = d\alpha_V$. We define $\delta[\alpha] = [\beta]$.

15 Poincaré Lemma

Definition 15.1 *A form α is* closed *if $d\alpha = 0$ and* exact *if $\alpha = d\eta$ for some η*

Since $d \circ d = 0$, exact forms are closed.

Definition 15.2 *The k-th* de Rham cohomology *of a manifold is the quotient of the space of closed k-forms by the space of exact k-forms.*

Theorem 15.3 *(Poincaré Lemma) If η is a closed k-form on $\mathbb{R}^n$ then it is exact.*

More generally if a manifold M is smoothly contractible to a point *and η is a closed form on M then it is exact.*

For example, a vector space and a ball are smoothly contractible to a point. A region $U \subset \mathbb{R}^n$ is smoothly contractible to a point if it is *star-shaped* (in other words there is $p_0 \in U$ s.t. $\forall p \in U$, $p_0 + t(p - p_0) \subset U$ for $0 \leq t \leq 1$).

Example 15.4 *If ω is the 1-form on $\mathbb{R}^2 \setminus \{0\}$ given by $\omega = \frac{xdy - ydx}{x^2+y^2}$, then $d\omega = 0$ but $\int_{S^1} \omega = 2\pi$. So ω is not exact, since the integral of an exact form around S^1 would be 0.*

Example 15.5 *The form*

$$\omega = \frac{x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}$$

on $\mathbb{R}^3 \setminus \{0\}$ is closed. It satisfies that its restriction to S^2 is the volume form on S^2. So $\int_{S^2} i^*\omega = 4\pi$.

15.1 Chain homotopy

(Spivak, Chapter 7)

Suppose M is a smooth manifold and $\imath_t : M \to M \times [0,1]$ is given by $\imath_t(p) = (p, t)$. Define $\mathcal{I} : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M)$ as follows. Write $\omega = \omega_1 + dt \wedge \eta$ where (for $\pi_M : M \times [0,1] \to M$) we have

1. $\omega_1(v_1, \ldots, v_k) = 0$ if some $v_i \sim \partial/\partial t$ (in other words if some $v_i \in \mathrm{Ker}(d\pi_M)$)

2. η is a $(k-1)$-form with this property

Define

$$\mathcal{I}\omega(p)(v_1,\ldots,v_{k-1}) = \int_0^1 \eta(p,t)((\imath_t)_* v_1,\ldots,(\imath_t)_* v_{k-1}).$$

Claim $\imath_1^*\omega - \imath_0^*\omega = d\mathcal{I}\omega + \mathcal{I}d\omega$.
Proof:

- Case 1: Assume first local coordinates $(x_1,\ldots,x_n)$ on a chart in M.

$$\omega = f(x,t)dx_{j_1}\wedge\ldots\wedge dx_{j_k}$$

(we denote the above by $f(x,t)dx_J$). Hence $d\omega$ is the sum of the term not involving dt plus $\frac{\partial f}{\partial t}dt\wedge dx_J$. So

$$\mathcal{I}(d\omega)(p) = \left(\int_0^1 \frac{\partial f}{\partial t'}(p,t')dt'\right) dx_J(p)$$

for $p\in M$

$$= \big(f(p,1)-f(p,0)\big)dx_J(p)$$
$$= i_1^*\omega(p) - i_0^*\omega(p)$$

and $\mathcal{I}\omega = 0$. So in this case $\mathcal{I}d\omega + d\mathcal{I}\omega = i_1^*\omega - i_0^*\omega$.

- Case 2: Assume $\omega = f(x,t)dt\wedge dx_J$. Then $i_1^*\omega = i_0^*\omega = 0$ because $i_1^*(dt) = d(c) = 0$ (where c is the constant function with value 1) since $i_1(m) = (m,1)$. Now

$$\mathcal{I}(d\omega)(p) = \mathcal{I}\big(-\sum_{\alpha=1}^n \frac{\partial f}{\partial x_\alpha} dt\wedge dx_\alpha\wedge dx_J\big)(p)$$

$$= -\sum_{\alpha=1}^n \left(\int_0^1 \frac{\partial f}{\partial x_\alpha}(p,t')dt'\right) dx_\alpha\wedge dx_J.$$

while

$$d(\mathcal{I}\omega) = d\left(\int_0^1 f(p,t')dt'\right) dx_j$$

$$= \sum_{\alpha=1}^n \frac{\partial}{\partial x_\alpha}\left(\int_0^1 f(p,t')dt'\right) dx_\alpha\wedge dx_J$$

$$= \sum_{\alpha=1}^n \left(\int_0^1 \frac{\partial f}{\partial x_\alpha}(p,t')dt'\right) dx_\alpha\wedge dx_J$$

$$= -\mathcal{I}(d\omega)$$

so in this case also $\mathcal{I}d\omega + d\mathcal{I}\omega = 0$.

Consequences:

1. If M is smoothly contractible to a point, then there exists a homotopy $H : M \times [0,1] \to M$ for which $H \circ i_1 : M \to M$ is the identity and $H \circ i_0 : M \to M$ is the constant map to a point p_0. Then $\omega = (H \circ i_1)^* \omega$ and $0 = (H \circ i_0)^* \omega$. We have

$$\omega = i_1^* H^* \omega - i_0^* H^* \omega$$

so if ω is closed, then $H^*\omega$ is closed and $i_1^* H^* \omega = \omega$ and $i_0^* H^* \omega = 0$. This shows $\omega = d(\mathcal{I} H^* \omega)$, so ω is exact. This gives a proof of the Poincaré lemma which works if M is contractible.

Definition 15.6 *If $F_0.F_1 : M \to N$ are smooth maps, then a homotopy between F_0 and F_1 is a smooth map $H : M \times [0,1] \to N$ for which $H \circ i_0 = F_0$ and $H \circ i_1 = F_1$. Here $i_0 : m \mapsto (m,0)$ and $i_1 : m \mapsto (m,1)$*

Consequences of chain homotopy $\mathcal{I}$:

If $H : M \times [0,1] \to N$ is a homotopy between F_0 and F_1, then $F_0^*\omega - F_1^*\omega = i_0^* H^*\omega - i_1^* H^* \omega = (d\mathcal{I} + \mathcal{I}d) H^*\omega = d(\mathcal{I}H^*\omega) + \mathcal{I}H^*(d\omega)$. So if $d\omega = 0$, then $[F_0^*\omega] = [F_1^*\omega]$.

Consequences of Poincaré lemma:

Cohomology of spheres

$H^\ell(S^k) \cong H^{\ell-1}(S^{k-1})$ for $k > 1$ and $\ell > 1$

Proof: (Sketch) (following Guillemin-Pollack p. 182) We will show that $H^k(S^k) = H^0(S^k) \cong \mathbb{R}$ and all the other groups are 0. $S^k = U_1 \cup U_2$ where $U_1 = \{(x_0, \ldots, x_k) : x_0 > -\epsilon\}$ and $U_2 = \{(x_0, \ldots, x_k) : x_0 < \epsilon\}$. So $U_1 \cap U_2 = \{U_2 = \{(x_0, \ldots, x_k) : -\epsilon < x_0 < \epsilon\}$

Recall that if F_0 and F_1 are homotopic then $F_0^*\omega - F_1^*\omega = (d\mathcal{I} + \mathcal{I}d)H^*\omega$ (where $\mathcal{I}$ is a chain homotopy $\omega^k(M \times [0,1]) \to \omega^{k-1}(M)$. Hence if $d\omega = 0$, $[F_0^*\omega] = [F_1^*\omega]$

Definition 15.7 *Two manifolds A and B are homotopy equivalent if there are maps $F : A \to B$ and $G : B \to A$ for which $F \circ G \simeq \mathrm{id}_B$ and $G \circ F \simeq \mathrm{id}_A$.*

Proposition 15.8 *If two manifolds A and B are homotopy equivalent then $H^*(A) \cong H^*(B)$.*

Proof:

$$F^* \circ G^* = \mathrm{id}_{H^*(A)},$$

$$G^* \circ F^* = \mathrm{id}_{H^*(B)},$$

(by the previous Proposition).

Definition 15.9 *If A is a manifold and B a submanifold, a deformation retraction from A to B is a map $r : A \to B$ where $i : B \to A$ is the inclusion map, for which $i \circ r : A \to A$ is the identity map and $r \circ i : B \to B$ is homotopic to the identity map. So if there is a deformation retraction from A to B then A and B are homotopy equivalent and $H^*(A) \cong H^*(B)$.*

Lemma 15.10 *There is a deformation retraction $r : A \to B$ where $A = U_1 \cap U_2$ and $B = S^{k-1}$ and*

$$r(x_0, \ldots, x_n) = \frac{(0, x_1, \ldots, x_n)}{\sqrt{x_1^2 + \ldots + x_n^2}}.$$

Proof:

$$H(t, (x_0, x_1, \ldots, x_n)) = \frac{((1-t)x_0, x_1, \ldots, x_n)}{||((1-t)x_0, x_1, \ldots, x_n)||}.$$

So $H \circ i_0 = \mathrm{id}$ and $H \circ i_1 = r \circ i$.

Hence $H^p(S^{k-1}) \cong H^p(U_1 \cap U_2)$.

Given a closed ℓ-form ω on $U_1 \cup U_2$, we produce a closed $(\ell-1)$-form η on $U_1 \cap U_2$. Start with ω. The restriction of ω to U_1 is exact (by the Poincaré lemma, since U_1 is smoothly contractible to a point).

$$i_{U_1}^* \omega = d\phi_1$$

and

$$i_{U_2}^* \omega = d\phi_2$$

for $(\ell-1)$-forms ϕ_1 on U_1 and ϕ_2 on U_2. Thus $d(\phi_1 - \phi_2) = 0$ on $U_1 \cap U_2$, so $\phi_1 - \phi_2 = \beta$ is a closed $(\ell-1)$-form on $U_1 \cap U_2$.

Given a closed $(\ell-1)$-form η on $U_1 \cap U_2$, we produce a closed ℓ-form on $U_1 \cup U_2$. Start with smooth functions ρ_1 on U_1 and ρ_2 on U_2 such that $\rho_1 = 0$ on a neighbourhood of the north pole N, and $\rho_2 = 0$ on a neighbourhood of the south pole S. (In fact we can assume $\rho_2 = 0$ on $U_2 \setminus (U_1 \cap U_2)$.) $\rho_i(x) \in [0, 1]$ and $\rho_1 + \rho_2 = 1$ everywhere. Thus $\rho_1 \beta$ is a form on U_1 and $\rho_2 \beta$ is a form on U_2 and we can define $\phi_1 = \rho_1 \beta$ and $\phi_2 = -\rho_2 \beta$. Note that $\phi_1 - \phi_2 = \beta$ on $U_1 \cap U_2$ as $\rho_1 + \rho_2 = 1$. Then $d\phi_1 - d\phi_2 = 0$ on $U_1 \cap U_2$ as $d\beta = 0$. Define $\omega \in \Omega^\ell(U_1 \cup U_2)$ by $\omega|_{U_1} = d\phi_1$ and $\omega|_{U_2} = d\phi_2$, and $d\omega = 0$ since this is true on U_1 and U_2. In fact these two procedures are inverse to each other (as indicated by the notation). So $H^\ell(S^k) \cong H^{\ell-1}(S^{k-1})$ for $\ell \geq 1$. Hence $H^k(S^k) \cong H^1(S^1) = \mathbb{R}$ and $H^0(S^k) \cong H^0(S^1) = \mathbb{R}$ and all the other groups are 0.

To see this, we use Mayer-Vietoris.

$$0 \longrightarrow \Omega^j(U_1 \cup U_2) \longrightarrow \Omega^j(U_1) \oplus \Omega^j(U_2) \longrightarrow \Omega^j(U_1 \cap U_2) \longrightarrow 0$$

This gives the long exact sequence

$$\ldots \longrightarrow H^j(U_1 \cup U_2) \longrightarrow H^j(U_1) \oplus H^j(U_2) \longrightarrow H^j(U_1 \cap U_2) \longrightarrow \ldots$$

Using this we can show the following theorem:

Theorem 15.11 $H^\ell(S^k) \cong H^{\ell-1}(S^{k-1})$.

16 Brouwer fixed point theorem

Theorem 16.1 *(Brouwer fixed point theorem) Any smooth map f from D^n to itself has a fixed point.*

Remark 16.2 *This theorem can be generalized to continuous maps.*

Proof: Suppose not. Then there is a map from D^n to S^{n-1} which is the identity on S^{n-1}, in other words there is a deformation retraction from D^n to S^{n-1}. We construct this map by using a line from x to $f(x)$ and checking where this line hits S^{n-1}. Define

$$g_t(x) = tx + (1-t)f(x) = f(x) + t(x - f(x)).$$

We solve

$$1 = t^2(x - f(x))^2 + 2tf(x)(x - f(x)) + f(x)^2$$

We take the solution t_0 with $t \geq 1$. This solution is a smooth function of x (using the quadratic formula). The map we seek is then $g_{t_0}(x)$. We note that $g_{t_0}(x)$ is a smooth function of x.

If $r : D^n \to S^{n-1}$ is a retraction, then $r \circ i = \mathrm{id}$ for $i : S^{n-1} \to D^n$ the inclusion. Then we have $i^* r^* = \mathrm{id}$ but this factors through $H^{n-1}(D^n) = \{0\}$. This is a contradiction.

17 Degree

In this section let M and N be compact oriented manifolds and $f : M \to N$ a smooth map.

Definition 17.1 *If M is connected and oriented, and $\dim(M) = n$, then*

$$H^n(M) = \Omega^n(M)/d\Omega^{n-1}(M).$$

(Note that for any $\alpha \in \Omega^n(M)$, $d\alpha = 0$.)

Proposition 17.2 *If M is compact, connected and oriented and $\partial M = \emptyset$, then there is a linear isomorphism $B : H^n(M) \to \mathbb{R}$ given by $B(\alpha) = \int_M \alpha$.*

Proof: Using a partition of unity we can construct an n-form α on M with $\int_M \alpha \neq 0$. If $\alpha = d\beta$ then $\int_M \alpha = 0$ by Stokes.

Remark 17.3 *Later we will give the proof that the map R given by $R([\alpha]) = \int_M \alpha$ is an isomorphism.*

An application of this result is the definition of degree.

Lemma 17.4 *There exists $\lambda \in \mathbb{R}$ for which $\int_M f^*\alpha = \lambda \int_N \alpha$ for all α. (λ depends only on f – it is independent of α.)*

Proof: f^* gives a linear map

$$\begin{array}{ccc} H^n(N) & \xrightarrow{f^*} & H^n(M) \\ \Big\downarrow B_N & & \Big\downarrow B_M \\ \mathbb{R} & \xrightarrow[\cdot\lambda]{} & \mathbb{R} \end{array}$$

λ is called the *degree* $\deg(f)$ of f.

Theorem 17.5 *If b is a regular value of f then $\deg(f) = n_+ - n_-$ where n_+ is the number of $p \in f^{-1}(b)$ for which $df_p : T_pM \to T_bN$ preserves orientation, while n_- is the number of $p \in f^{-1}(b)$ for which $df_p : T_pM \to T_bN$ reverses orientation.*

Proof: $f^{-1}(q)$ is a finite set of points $p_1, \ldots, p_k$. Choose a neighbourhood U_i around each p_i on which $f|_{U_i}$ is a diffeomorphism. Such a neighbourhood exists because df_{p_i} is an isomorphism for all p_i. In particular these neighbourhoods cannot intersect.

Choose a compact neighbourhood W around q, and define $W' := f^{-1}(W) \setminus \coprod_j U_j$. Choose a neighbourhood V of q in W, so $f^{-1}(V) \subset U_1 \cup \ldots \cup U_k$. Redefine U_i to be $U_i \cap f^{-1}(V)$, so $f : U_i \to V$ is a diffeomorphism. Choose $\omega = g dy_1 \wedge \ldots \wedge dy_n$ where $g \geq 0$ has compact support contained in V. Then the support of $f^*\omega$ is contained in $U_1 \cup \ldots \cup U_k$. So $\int_M f^*\omega = \sum_{j=1}^k \int_{U_j} f^*\omega$. But since $f : U_i \to V$ is a diffeomorphism, $\int_{U_i} f^*\omega = \int_V \omega$ if f is orientation preserving on U_i, while $\int_{U_i} f^*\omega = -\int_V \omega$ if f is orientation reversing on U_i.

Remark 17.6 *The degree of f is independent of the choice of regular value. So since b with $f^{-1}(b) = \emptyset$ is a regular value, the degree of f is 0 unless f is surjective.*

Example 17.7 *If M and N are compact oriented manifolds and $F : M \to N$ is an orientation-preserving covering map with n sheets (for example $F : S^1 \to S^1$ defined by $F(e^{i\theta}) = e^{in\theta}$) then $\int_M F^*\omega = n \int_N \omega$ (as a consequence of the previous theorem).*

17.1 Consequences of chain homotopy

Proposition 17.8 *Homotopic maps have the same degree.*

Proof: If $H : M \times [0,1] \to N$ is a homotopy between the maps $F_0 : M \to N$ and $F_1 : M \to N$, then $F_0 = H \circ i_0$ and $F_1 = H \circ i_1$ so $F_0^*\omega - F_1^*\omega = i_0^* H^*\omega - i_1^* H^*\omega = d(\mathcal{I}H^*\omega) + \mathcal{I}dH^*\omega$. Hence if M and N are compact oriented manifolds and F_0 and F_1 are homotopic then $\int_M F_0^*\omega = \deg(F_0) \int_N \omega$ and $\int_M F_1^*\omega = \deg(F_1) \int_N \omega$. But by Stokes' theorem $\int_M F_1^*\omega - \int_M F_0^*\omega = \int_M d\mathcal{I}H^*\omega$ (recall that ω is closed since it is an m-form on an m-dimensional manifold) so $\deg F_0 = \deg F_1$.

17.2 Consequences of degree

Proposition 17.9 *"Hairy ball theorem") If n is even, there is no nowhere zero vector field on S^n.*

Proof: Let A be the antipodal map. Then A is an orientation reversing diffeomorphism since n is even (it is an orientation preserving diffeomorphism when n is odd). Because a reflection is an orientation reversing diffeomorphism, it has degree -1. The antipodal map of S^n is the composition of $n+1$ reflections, so $deg(A) = (-1)^{n+1}$ (see Proposition 17.11 below). At the same time the degree of the identity map is 1. But if there is a nowhere zero vector field X on

S^n then we can construct a homotopy between A and the identity map. For each p, there is a unique great semicircle γ_p from p to $A(p) = -p$ whose tangent vector at p is a multiple of $X(p)$ (if p is the north pole, γ_p would be the longitude whose tangent at p is is $X(p)$).

Note that for n odd we *can* construct a nowhere zero vector field on S^n: for $p = (x_0, \dots, x_n) \in S^n$ we define

$$X(p) = (-x_1, x_0, -x_3, x_2, \dots, -x_{n+1}, x_n) = (-x_1\partial_{x_0} + x_0\partial_{x_1} + \dots + (-x_{n-1}\partial_{x_n} + x_n\partial_{x_{n-1}})$$

Theorem 17.10 *If M is a compact orientable k-manifold then $H^k(M) \cong \mathbb{R}$.*

The isomorphism is given by the map integrating differential forms over M.
Proof:

- Step 1: $H^k(S^k) \cong \mathbb{R}$, and the isomorphism is given by $\omega \mapsto \int \omega$. (This was already proved, since we proved $H^k(S^k) \cong H^{k-1}(S^{k-1})$ and $H^1(S^1) \cong \mathbb{R}$)

- Step 2: If the k-form ω is compactly supported in $\mathbb{R}^k$, $\omega = d\eta$ for some compactly supported η iff $\int_{\mathbb{R}^k} \omega = 0$. (Proof: Let $\Phi : S^k \setminus \{N\} \to \mathbb{R}^k$ be the stereographic projection. Then $\Phi^*\omega = \omega'$ is a differential form on S^k, for some ω' such that $\omega = 0$ on a contractible neighbourhood U of N. So if $\int_{\mathbb{R}^k} \omega = 0$ then $\int_{S^k} \Phi^*\omega = 0$ so $\omega' = d\nu$ on S^k. As $d\nu = 0$ on the contractible neighbourhood U of N, $i_U : U \to S^k$ and $d(i_U^*\nu) = 0$. By the Poincaré lemma, $i_U^*\nu = d\mu$ for a $(k-2)$-form μ on U. Hence $i_U^*\nu - d\mu = 0$ and $\gamma := (\Phi^{-1})^*(i_U^*\nu - d\mu)$ is defined on $\mathbb{R}^k$ and compactly supported, and $\omega = d\gamma$.

  If $\omega = d\eta$ and η is compactly supported, then we choose R so large that $Supp(\eta) \subset \{x \in \mathbb{R}^k | |x| \leq R\} := B_K(R)$. Then $\int_{\mathbb{R}^k} \omega = \int_{B_K(R)} \omega = \int_{B_K(R)} d\eta = \int_{\partial B_K(R)} \eta = 0$.

- Step 3: If $U \subset M$ is an open set diffeomorphic to $\mathbb{R}^k$, then any k-form β compactly supported in U satisfies

  $$[\beta] = (\int_U \beta)[\omega]$$

  for any k-form ω compactly supported in U with $\int_U \omega = 1$. (This follows from Step 2, because $\alpha := \beta - (\int_U \beta)\omega = d\eta$ for some η, because $\int_U \alpha = 0$.)

- Step 4: Cover M by a finite number of open sets $U_1, \dots, U_N$ and pick homotopies $H_i : M \times I \to M$ with $H_i|_{M\times\{0\}} = \text{id}$ and $H_i|_{M\times\{1\}} = G_i : U_i \to U$. So $[\omega] = G_i^*[\omega]$ if ω is the extension to M of a form compactly supported on U with $\int_U \omega = 1$. This is true because homotopic maps induce the same map in de Rham cohomology.

- Step 5: Choose a partition of unity $\{f_i\}$ subordinate to $\{U_i\}$. Any closed k-form θ equals $\sum_i f_i\theta$, where $f_i\theta$ is supported in U_i and

$$[f_i\theta] = \int_{U_i} f_i\theta \cdot [G_i^*\omega]$$

(by Step 3, because $G_i^*\omega$ is compactly supported on U_i with $\int_{U_i} G_i^*\omega = 1$ – in Step 3, we replace β by $f_i^*\theta$ and replace ω by $G_i^*\omega$)

$$= (\int_{U_i} f_i\theta) \cdot [\omega]$$

(by Step 4). Hence

$$[\theta] = \sum_i [f_i\theta] = \sum_i \int_M f_i\theta \cdot [\omega]$$

$$= (\int_M \theta) \cdot [\omega],$$

in other words $[\theta] = 0$ iff $\int_M \theta = 0$.

This completes the proof that $F : H^n(M) \to \mathbb{R}$ given by $F(\alpha) = \int_M \alpha$ is an isomorphism.

17.3 Further results about the degree

Proposition 17.11 *if $f : M \to N$ and $g : N \to P$ then*

$$deg(fg) = deg(f) \cdot (deg(g)$$

Proof: This follows from Theorem 17.5.

Proposition 17.12 *The degree of an orientation reversing diffeomorphism is -1. Hence an orientation reversing diffeomorphism cannot be homotopy equivalent to the identity map.*

Proof: This also follows from Theorem 17.5.

Proposition 17.13 *The antipodal map of S^n is the composition of $n+1$ reflections, so its degree is $(-1)^{n+1}$. So if n is even, the antipodal map is not homotopy equivalent to the identity. If n is odd, the antipodal map is homotopy equivalent to the identity (via the flow of a nowhere zero vector field on S^n).*

Proposition 17.14 *If $f : S^n \to S^n$ has no fixed points, then deg$(f) = (-1)^{n+1}$.*

Proof: If $f(x) \neq x$ for any x, then the line

$$t \mapsto (1-t)f(x) + t(-x), \quad 0 \leq t \leq 1$$

does not pass through 0. So

$$f_t(x) = \frac{(1-t)f(x) - tx}{|\ (1-t)f(x) - tx\ |}$$

defines a homotopy from from f to the antipodal map, which has degree $(-1)^{n+1}$.)

17.4 Applications of degree to group actions

Definition 17.15 *A group G acts on a space X if there is a homomorphism $G \mapsto \text{Homeo}(X)$.*

Definition 17.16 *A group acts freely on X if the homeomorphism corresponding to each nontrivial element of G has no fixed points.*

Example 17.17 *The rotation group $SO(3)$ acts on $\mathbb{R}^3$ and S^2; the group $U(1)$ acts on $\mathbb{R}^2 = \mathbb{C}$ by*

$$e^{i\theta} : z \mapsto e^{i\theta} z.$$

Proposition 17.18 *If n is even, then $\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n.*

Remark 17.19 *Note that S^1 and S^3 are groups (see the section on Lie groups) and all Lie groups act freely on themselves (by left or right multiplication) so there are some odd dimensional spheres which admit a free action of a group other than $\mathbb{Z}_2$.*

Proof: The degree of a homeomorphism must be ± 1. Hence a group action determines a function $D : G \to \{\pm 1\}$ which is a homomorphism (by Proposition 17.11 above). If the action is free, D sends every nontrivial element of G to $(-1)^{n+1}$ (using Proposition 17.14). So if n is even, D sends all the nontrivial elements of G to -1. Hence

$$Ker(D) = \{1\}.$$

So since

$$D : G/\text{Ker(D)} \cong \{\pm 1\},$$

we learn that $G \cong \{\pm 1\} = \mathbb{Z}_2$.

18 Riemannian metrics

Definition 18.1 *A Riemannian metric is an assignment of an inner product on T_xM for all $x \in M$, such that $g_{ij} := g(\partial_i, \partial_j)$ are smooth functions.*

Definition 18.2 *The length of a curve $\gamma : [a, b] \to M$ is*

$$\int_a^b g_{\gamma(t)}(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2} dt.$$

Proposition 18.3 *The length of a curve γ is independent of the parametrization of γ.*

Proof:

$$\int_a^b g_{\gamma(s)}(\frac{d\gamma}{ds}, \frac{d\gamma}{ds})^{1/2} ds = \int_a^b g_{\gamma(t)} \left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \cdot \left(\frac{dt}{ds} \right)^2 \right)^{1/2} \frac{ds}{dt} dt = \int_a^b g_{\gamma(t)}(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2} dt$$

Definition 18.4 *The arc length of the curve γ is $L(t) = \int_a^t g_{\gamma(s)}(\frac{d\gamma}{ds}, \frac{d\gamma}{ds})^{1/2} ds$*

Definition 18.5 *The volume element is*

$$\omega = \sqrt{\det g} dx_1 \wedge \ldots \wedge dx_n$$

where $\det g$ refers to the determinant of the $n \times n$ matrix g_{ij}.

The volume element equals $w^1 \wedge \ldots \wedge w^n$ if $\{w^j\}$ is the basis for T_x^*M dual to an orthonormal basis $\{e_i\}$ of T_xM. The Riemannian volume element is $\sqrt{\det h} w^1 \wedge \ldots \wedge w^n$ if $\{w^i\} \in \Gamma(T^*M)$ is the dual basis to a basis of vector fieldsx $\{X_i\}$. h is the matrix $h_{ij} = g(X_i, X_j)$. For example we might write $X_i = \frac{\partial}{\partial x^i}$.

Proposition 18.6 *The Riemannian volume element is independent of the choice of basis $\{X_i\}$.*

Proof: Let $\{Z_i\} \in \Gamma(TM)$ be another basis of tangent vectors and let $\{\eta^i\} \in \Gamma(T^*M)$ its dual basis. Put $f_{ij} = g(Z_i, Z_j)$ and define a matrix γ by $X_j = \sum_\ell Z_\ell \gamma_{\ell j}$. The determinant of γ must be positive because both $\{X_j\}$ and $\{Z_j\}$ are compatible with the orientation.

Then

$$h_{ij} = g(X_i, X_j) = \sum_{\ell,m} \gamma_{\ell i} \gamma_{mj} g(Z_\ell, Z_m)$$

$$= \sum_{\ell,m} \gamma_{\ell i} \gamma_{mj} f_{\ell m}.$$

So $h = \gamma^T f \gamma$, and $\sqrt{\det h} = \det(\gamma)\sqrt{\det f}$. Thus evaluating on $(X_1, \ldots, X_n)$ we see that

$$\eta^1 \wedge \ldots \wedge \eta^n = (\det \gamma)\omega^1 \wedge \ldots \wedge \omega^n$$

and so

$$\sqrt{h}\omega^1 \wedge \ldots \wedge \omega^n = \sqrt{\det f}\eta^1 \wedge \ldots \wedge \eta^n.$$

The collection of vector fields on M will be denoted $\Xi(M)$.

Definition 18.7 *A* connection *on TM is a map $\nabla : \Xi(M) \times \Xi(M) \to \Xi(M)$ denoted $(X, Y) \mapsto \nabla_X Y$ for which*

1. *$\nabla_{fX} Y = f\nabla_X Y$ for f a smooth function and X, Y vector fields on M.*
2. *$\nabla_X(fY) = (Xf)Y + f\nabla_X Y$.*

More generally if V is a vector bundle over M, $\nabla : \Xi(M) \times \Gamma(V) \to \Gamma(V)$ where $X \in \Xi(M)$ and $Y \in \Gamma(V)$.

Note that if $M = \mathbb{R}^n$ (for smooth functions f_j) there is an obvious connection $\nabla_{\partial_i}(\sum_j f_j \partial_j) = \sum_j (\partial_i f_j)\partial_j$ but this depends on the coordinate choice. A connection provides a way to compare tangent vectors at different points in a manifold.

Proposition 18.8 *For any connection on TM, write $\xi_i = \partial_i$ and $\nabla_{\xi_i}\xi_j = \sum_{k=1}^n \Gamma_{ij}^k \xi_k$. The Γ_{ij}^k are called Christoffel symbols.*

Definition 18.9 *The torsion of a connection is*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

We can compute that the torsion is a tensor, in other words if we multiply X or Y by a smooth function f , $T(fX, Y) = fT_X Y$ and $T(X, fY) = fT_X Y$ – there is no dependence on derivatives of f.

Proposition 18.10 *∇ is torsion free iff in local coordinates $\Gamma_{ij}^k = \Gamma_{ji}^k$*

Definition 18.11 *(Connection along a curve, or covariant derivative) Let $\Xi(s)$ be the collection of smooth maps $v : [a, b] \to TM$ such that $v(s(t)) \in T_{s(t)}M$. The covariant derivative associated to a connection ∇ is $\nabla/dt : \Xi(s) \to \Xi(s)$ such that*

1. *∇/dt is $\mathbb{R}$-linear*
2. *$\nabla/dt(fv) = (df/dt)v + f(\nabla/dtv)$*

3. If $Y \in \Xi(M)$ then $\nabla/dt(Y|_s) = \nabla_{\dot{s}(t)}(Y)$.

Definition 18.12 *A vector field $v \in \Xi(s)$ is* parallel *along s if $(\nabla/dt)v = 0$.*

Definition 18.13 *For a curve s in a submanifold $M \subset \mathbb{R}^n$ and a vector field Y along the curve s, we define a covariant derivative by $((\nabla/dt)Y)(t) = \pi(dY/dt)$, where π is the projection from $\mathbb{R}^n$ onto $T_{s(t)}M$.*

Define $< X, Y >:= g(X, Y)$

Proposition 18.14 *$d/dt < v, w >=< \nabla/dtv, w > + < v, \nabla/dtw >$ In particular if v and w are vector fields parallel along a curve s, relative to the Levi-Civita connection, they make a constant angle with each other and have constant lengths along s.*

Proof: If $v = \sum v^i \xi_i$, $w = \sum_i w^i \xi_i$, in terms of $\xi_i = \partial_i$ we have $\nabla v/dt = \sum_k dv^k/dt\xi_k + v^k \nabla_{\dot{s}} \xi_k$. The right hand side is

$$\sum < dv^k/dt\xi_k, w^\ell \xi_\ell > + < v^k \xi_k, dw^\ell/dt\xi_\ell > + \sum < v^k \nabla_{\dot{s}} \xi_k, w^\ell \xi_\ell > + \sum < v^k \xi_k, w^\ell \nabla_{\dot{s}} \xi_\ell >$$

This is the sum of the first two terms plus $\sum v^k w^\ell \dot{s} < \xi_k, \xi_\ell >$ (because ∇ is Riemannian) $= \frac{d}{dt} < v, w >$.

Theorem 18.15

$$< \nabla_X Y, Z > - < \nabla_Y X, Z >=< [X, Y], Z >$$

Poincaré Duality and the Hodge Star Operator

Let M be a compact oriented manifold of dimension n.

Definition 18.16 *The Hodge star operator is a linear map*

$$* : \Omega^k(M) \to \Omega^{n-k}(M)$$

which satisfies

- $$* \circ * = (-1)^{k(n-k)}$$

- $$\alpha \wedge *\alpha = |\alpha|^2 vol$$

  where vol is the standard volume form and $|\alpha|^2$ is the usual norm on $\alpha(x)$ viewed as an element of $\Lambda^k T_x^* M$.

The definition of the Hodge star operator requires the choice of a Riemannian metric on the tangent bundle to M.

Let d be the exterior differential. Then $d^* := *d*$ is the formal adjoint of d, in the sense that $(d^*a, b) = (a, db)$. This is because $(*a, *b) = (a, b)$ for any $a, b \in \Omega^k M$, so

$$(da, b) = \int da^* b = (-1)^k \int a^d * b$$

(by Stokes' theorem)

$$= (-1)^{k(n-k)}(-1)^k (a, *d * b)$$

Definition 18.17 *A k-form α on M is* harmonic *if $d\alpha = d^*\alpha = 0$.*

Theorem 18.18 *The set of harmonic k-forms is isomorphic to $H^k(M; \mathbb{R})$.*

Theorem 18.19 *If α is a harmonic k-form on M, its Poincare dual is represented by $*\alpha$. The pairing between an element α and its Poincare dual is nondegenerate, i.e. for any form α $\int_M \alpha \wedge *\alpha = 0 \longrightarrow \alpha = 0$.*

19 Lie Groups

Definition 19.1 *A Lie group is a group G which is also a smooth manifold, for which multiplication $m : G \times G \to G$ and inversion $i : G \to G$ are smooth maps. The identity element is usually denoted e.*

Example 19.2 $U(1)$*:* $m(e^{i\sigma}, e^{i\tau}) = e^{i(\sigma+\tau)}$ $i(e^{i\sigma}) = e^{-i\sigma}$.

Example 19.3 $GL(n,\mathbb{R}) : m(A,B)_{ij} = \sum_r A_{ir}B_{rj}$ $i(A)_{ji} = \frac{(-1)^{i+j}}{\det A}\tilde{A}_{ij}$ *where $\tilde{A}_{ij}$ is the determinant of the matrix obtained by striking out the i-th row and j-th column of A.*

Example 19.4 $GL(n,\mathbb{C})$*: the definition is exactly the same as $GL(n,\mathbb{R})$ with $\mathbb{R}$ replaced by $\mathbb{C}$*

Example 19.5 $\mathbb{R}$*:* $m(a,b) = a+b$, $i(a) = -a$

Example 19.6 $O(n) = \{A \in M_{n\times n}(\mathbb{R}) : AA^T = 1\}$ *where 1 is the $n \times n$ identity matrix.*

Example 19.7 $SO(n) = \{A \in O(n) : \det A = 1\}$

Example 19.8 $U(n) = \{A \in GL(n, \mathbb{C}" AA^\dagger = 1\}$ *where* $A^\dagger = \bar{A}^T$

Example 19.9 $SU(n) = \{A \in U(n) : \det A = 1\}$

Definition 19.10 *A* Lie subgroup *of G is a regular submanifold which is also a subgroup of G.*

Lie subgroups are necessarily Lie groups, with their smooth structure as submanifolds of G. The multiplication and inversion maps are automatically smooth. Lie subgroups are necessarily closed (Boothby, Theorem III.6.18).

Example 19.11

1. $O(n) = \{A \in GL(n,\mathbb{R}) : AA^T = 1\}$ *is a Lie subgroup of $GL(n,\mathbb{R})$.*
2. $SO(n) = \{A \in O(n) : \det(A) = 1\}$ *is a Lie subgroup of $GL(n,\mathbb{R})$.*
3. $U(n) = \{A \in GL(n,\mathbb{C}) : AA^\dagger = 1\}$ *is a Lie subgroup of $GL(n,\mathbb{C})$. (Here $A^\dagger$ is the conjugate of the transpose of A.)*

Example 19.12 *(Sp(n)) The group*

$$Sp(n) = \{M(A,B) := \begin{bmatrix} A & -\bar{B} \\ B & \bar{A} \end{bmatrix}\}$$

where $A, B \in \mathrm{End}(\mathbb{C}^n)$ *and we insist that* $M(A,B) \in U(2n)$*. Equivalently*

$$Sp(n) = \{U \in SU(2n) : \bar{U}J = JU\}$$

where $J = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}$.

Classical groups:

- A_n ... $SU(n+1), n \geq 1$
- B_n ... $SO(2n+1), n \geq 2$
- C_n ... $Sp(n), n \geq 3$
- D_n ... $SO(2n), n \geq 4$

The reason for the restriction on n is to avoid duplication: for low values of n many of the groups are isomorphic, or at least their Lie algebras are. For example $SO(3)$ has the same Lie algebra as $SU(2)$.

The classical groups above and a finite list of "exceptional Lie groups" (G_2, F_4, E_6, E_7, E_8) are basic building blocks for compact connected Lie groups.

Theorem 19.13 *If* G_1 *and* G_2 *are Lie groups and* $F : G_1 \to G_2$ *is a smooth map which is also a homomorphism, then* $\mathrm{Ker}(F)$ *is a closed regular submanifold which is a Lie group of dimension* $\dim(G_1) - \mathrm{rk}(F)$.

Example 19.14 $SL(n,\mathbb{R})$ *is the kernel of* $\det : GL(n, \mathbb{R} \to \mathbb{R} \setminus \{0\}$.

Proof: This is Boothby, , Theorem III.6.14.

Definition 19.15 *A Lie subgroup* H *of a Lie group* G *is a subgroup (algebraically) which is a submanifold and is a Lie group (with its smooth structure as an immersed submanifold).*

Proposition 19.16 *A Lie subgroup that is a regular submanifold is closed. Conversely a Lie subgroup that is closed is a regular submanifold.*

(Recall: $X \subset M$ is a regular submanifold iff there is a chart $\phi : U \to \mathbb{R}^m$ for which $\phi(U \cap X) = \phi(U) \cap \mathbb{R}^n$.)

Definition 19.17 *Let $F : N \to M$ be a diffeomorphism and X a vector field on N, while Y is a vector field on M. Then X is F-related to Y iff $F_*(X_m) = Y_{F(m)}$ for all $m \in M$.*

Proposition 19.18 *The Lie brackets of F-related vector fields are F-related.*

Proof: We have to show that if X_i, Y_i are F-related vector fields then

$$dF([X_1, X_2]) = [Y_1, Y_2].$$

We are assuming $dF(X_i) = Y_i$. For all $g \in C^\infty(V)$, and $x \in F^{-1}(V)$,

$$(Yg)(F(x)) = (dF)_x(X)(g) = X(g \circ F) \tag{2}$$

This is equivalent to

$$(Yg) \circ F = X(g \circ F).$$

If $f \in C^\infty(V)$, we replace g by $Y_2 f$, and Y by Y_1 in (2). This gives

$$Y_1(Y_2 f) \circ F = X_1((Y_2 f) \circ F).$$

Now apply (2) for $g = f$, $Y = Y_2$. This gives

$$Y_1(Y_2 f) \circ F = X_1(X_2(f \circ F)).$$

Likewise

$$Y_2(Y_1 f) \circ F = X_2(X_1(f \circ F)).$$

So

$$([Y_1, Y_2]f) \circ F = [X_1, X_2](f \circ F)$$

so $[Y_1, Y_2]$ is F-related to $[X_1, X_2]$.

19.1 Left invariant vector fields

For $g \in G$ define $L_g : G \to G$ by $L_g(h) = g \circ h$. For $Y \in T_eG$ define a vector field $\tilde{Y}$ by $\tilde{Y}_g = (L_g)_* Y$.

Proposition 19.19 *$\tilde{Y}$ is a smooth vector field.*

Proposition 19.20 *$[\tilde{X}, \tilde{Y}]$ is left invariant.*

Proof: For any $h \in G$, $\tilde{Y}$ is F-related to itself (where $F = L_h$), so $[\tilde{Y_1}, \tilde{Y_2}]$ is also F-related to itself, in other words it is left invariant (so $[\tilde{Y_1}, \tilde{Y_2}] = \tilde{Z}$ for some $Z \in T_eG$. Hence there is an operation $[\cdot, \cdot]$ on T_eG (*Lie bracket*). T_eG equipped with $[\cdot, \cdot]$ is called the Lie algebra of G, denoted Lie(G).

Proposition 19.21 *The tangent bundle TG of a Lie group G is trivial.*

Proof: We have a global basis of sections given by the left invariant vector fields.

Example 19.22 *TS^3 is trivial, since $S^3 = SU(2)$.*

Theorem 19.23 *For every $X \in T_eG$ there is a unique smooth homomorphism $\phi : \mathbb{R} \to G$ with $d\phi/dt|_{t=0} = X$.*

Proof: Given X, we construct the corresponding left invariant vector field $\tilde{X}$. Take the integral curve $\phi : (-\epsilon, \epsilon) \to G$ through e (with $\phi(0) = e$). Extend it to $\phi : \mathbb{R} \to G$ by defining

$$\phi(t) = \phi(\epsilon/2) \circ \ldots \phi(\epsilon/2)\phi(r)$$

where the number of $\phi(\epsilon/2)$ is k and $t = k(\epsilon/2) + r$. Then $t \mapsto \phi(s) \cdot \phi(t)$ is an integral curve of $\tilde{X}$ passing through $\phi(s)$ at $t = 0$. Also, $\phi(s + t)$ is such an integral curve. So by uniqueness of integral curves

$$\phi(s + t) = \phi(s) \cdot \phi(t).$$

Conversely if $\phi : \mathbb{R} \to G$ is a smooth homomorphism, and $f : G \to \mathbb{R}$ is smooth, then $d\phi/dt$ is a tangent vector to G at $\phi(t)$. Recall

$$\frac{d\phi}{dt}(f) = \lim_{h \to 0} \frac{f(\phi(t + h)) - f(\phi(t))}{h}$$

$$\lim_{h \to 0} \frac{f(\phi(t)\phi(h)) - f(\phi(t))}{h}$$

$$= \frac{d}{du} |_{u=0} \, f \circ L_{\phi(t)} \circ \phi(u)$$

$$= (L_{\phi(t)})_* \frac{d}{du} |_{u=0} \, (f)$$

$$= (L_{\phi(t)})_* X(f) = \tilde{X}(\phi(t))(f).$$

So ϕ is an integral curve of $\tilde{X}$.

Definition 19.24 *A one parameter subgroup of G is a homomorphism phi $: \mathbb{R} \to G$.*

We have thus shown that there is a bijective correspondence between left invariant vector fields and one parameter subgroups.

Given $X \in \mathrm{Lie}(G)$, let ϕ be the unique smooth homomorphism with $\frac{d\phi}{dt}(0) = X$. Then we define the exponential map as follows.

Definition 19.25 *(Exponential map) With the above notation,*

$$\exp(X) = \phi(1).$$

Clearly

$$\exp(t_1 + t_2)X = (\exp t_1 X)(\exp t_2 X)$$

and

$$\exp(-tX) = (\exp tX)^{-1}.$$

Proposition 19.26 *The map* $\exp : \mathrm{Lie}(G) \to G$ *is smooth, and* 0 *is a regular value so* $\exp$ *takes a neighbourhood of* $0 \in \mathrm{Lie}(G)$ *diffeomorphically onto a neighbourhood of* $e \in G$*.*

Proof: Define a vector field Y on $\mathrm{Lie}(G) \times G$ by

$$Y_{(X,a)} = 0 \oplus \tilde{X}(a).$$

(Note that $T_{(X,a)}(\mathrm{Lie}(G) \times G) \cong T_e G \oplus T_a G$.) Then Y has a flow

$$\alpha : \mathbb{R} \times (T_e G \times G) \to T_e G \times G$$

which is smooth (since Y is smooth). Since $\exp(X)$ is the projection on G of $\alpha(1, 0 \oplus X)$, exp is smooth (as it is the composition of smooth maps).

Given $v \in T_e G$, the curve $c(t) = tv$ in $T_e G$ has tangent vector v at 0.

So

$$\exp_0(v) = \frac{d}{dt}\,|_0 \exp(tv) = v.$$

Hence

$$(d\exp)\,|_0 = \mathrm{id}.$$

So exp is a diffeomorphism in a neighbourhood of 0.

Proposition 19.27 *If* $\psi : G \to H$ *is a homomorphism then*

$$\exp_H \circ d\psi = \psi \circ \exp_G.$$

Proof: If $\psi : G \to H$, and $X \in T_e G$, then let $\psi : \mathbb{R} \to G$ be a homomorphism with

$$\frac{d\phi}{dt} \mid_{t=0} = X.$$

Then $\psi \circ \phi : \mathbb{R} \to H$ is a homomorphism with

$$\frac{d}{dt}(\psi \circ \phi) \mid_{t=0} = \psi_* X.$$

So

$$\exp(\psi_(X) = \psi \circ \phi(1) = \psi(\exp X).$$

Proposition 19.28 *If* $G = GL(n, \mathbb{R})$ *then* $\mathrm{Lie}(G) = M_{n \times n}(\mathbb{R})$ *(the vector space of* $n \times n$ *real matrices) and*

$$\exp(X) = \sum_{n \geq 0} \frac{X^n}{n!}. \tag{3}$$

Proof: We define a norm on $\mathrm{Lie}(G)$ as follows:

$$|X| = \sup_{1 \leq i,j \leq n} |x_{ij}|$$

sp

$$|X^k| \leq \frac{1}{n}(n|X|)^k$$

(since $|AB| \leq n|A||B|$). Hence the series (3) converges absolutely. Also the one parameter subgroup of $GL(n, \mathbb{R})$ whose left invariant vector field has the value X at e is $\exp(tX)$ since

$$\sum_{n \geq 0} \frac{t^n X^n}{n!} = \mathrm{id} + tX + O(t^2)$$

hence

$$\frac{d}{dt} \mid_{t=0} \sum_{n \geq 0} \frac{t^n X^n}{n!} = X.$$

Proposition 19.29 *If* $G = GL(n, \mathbb{R})$ *and* $A, B \in \mathrm{Lie}(G)$ *then*

$$[A, B] = AB - BA.$$

Proof:

$$A = \sum_{i,j} a_{ij} \frac{\partial}{\partial x_{ij}},$$

$$B = \sum_{i,j} b_{ij} \frac{\partial}{\partial x_{ij}}$$

where a_{ij}, b_{ij} are constants. Let $\tilde{A}, \tilde{B}$ be the left invariant vector fields corresponding to A and B. Then

$$[\tilde{A}, \tilde{B}]f = \tilde{A}(\tilde{B}f) - \tilde{B}(\tilde{A}f_)$$

(by definition of the Lie bracket on vector fields).

If $x \in GL(n, \mathbb{R})$, then

$$\tilde{B}(x)_{ij} = (xB)_{ij} = \sum_r x_{ir} b_{rj}$$

so

$$A(\tilde{B}f) = \sum_{i,j} \sum_{k,\ell} a_{k\ell} \frac{\partial}{\partial x_{k\ell}}$$

$$= \sum_r a_{ir} b_{rj} \frac{\partial}{\partial x_{ij}} r + \text{terms} \quad \text{with} \ \frac{\partial}{\partial x_{k\ell}} \frac{\partial}{\partial x_{ij}}$$

Likewise

$$B(\tilde{A}f) = \sum_r b_{ir} a_{rj} \frac{\partial}{\partial x_{ij}} f.$$

It follows that

$$[\tilde{A}, \tilde{B}] = A\widetilde{B - B}A.$$

Proposition 19.30 *If* $[X, Y] = 0$ *then* $\exp(X + Y) = \exp X \exp Y$.

Proof: For matrix groups,

$$\exp(X + Y) = \sum_{n \geq 0} \frac{(X + Y)^n}{n!}$$

$$= \sum_{m=0}^{\infty} \sum_{p=0}^{m} \frac{1}{(m-p)!} X^{m-p} \frac{1}{p!} Y^p$$

$$= \Big(\sum_{k=0}^{\infty} \frac{1}{k!} X^k \Big) \Big(\sum_{\ell=0}^{\infty} \frac{1}{\ell!} X^\ell \Big)$$

$$= \exp X \exp Y.$$