A PROOF OF THEOREM 5.67 IN "CHAOTIC BILLIARDS" BY CHERNOV AND MARKARIAN

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Abstract. The proof of [2, Theorem 5.67] relies on an estimate which is left as an exercise (namely [2, Exercise 5.69]) but appears to be incorrect. In these notes we provide a proof of the above mentioned result.

1. The problem

This short note proposes a correction to the proof of [2, Theorem 5.67]. The theorem is further used in the cited book to prove the so-called *Fundamental Theorem* [2, Theorem 5.70] and it is also heavily relied upon in [1].

In this paper uwe will use the notation introduced in various stages in [2]. We will only explicitly recall those definitions that need to be corrected. Here follows the statement of the theorem that we will prove.

Theorem 5.67. *For any weakly homogeneous unstable curve (see* [2, Definition 5.9]*) $W \subset \mathcal{M}$, the stable H-manifold $W^s_{\mathbb{H}}(x)$ exists (i.e. $r^s_{\mathbb{H}} > 0$) for $\mathbf{m}_W$-almost every $x \in W$. Furthermore, there exists $C > 0$ so that:*

$$\mathbf{m}_W\{x \in W : r^s_{\mathbb{H}}(x) < \varepsilon\} \le C\varepsilon.$$

The proof of the above theorem outlined in [2] relies on [2, Exercise 5.69], which claims that there exist a constant $C > 0$ so that for any weakly homogeneous unstable curve W, any $x \in W$ and any $n \ge 1$:

$$CE_s(\mathcal{F}^n x)d^s(\mathcal{F}^n x, \mathcal{S}^{\mathbb{H}}_{-1}) \ge r_{n-1}(x), \tag{1.1}$$

where $E_s(y)$ bounds the expansion of stable curves by $\mathcal{F}^{-1}$ at y (see [2, (5.57)]), d^s denotes the distance *along stable curves* and, finally, for $x \in W$, we denote with $r_n(x)$ the distance (measured along the curve $\mathcal{F}^n W$) of $\mathcal{F}^n x$ from the boundary of the H-component of $\mathcal{F}^n W$ which contains $\mathcal{F}^n x$.

We observe that (1.1) does not hold: on the one hand $\mathcal{F}^{n-1}x$ may be arbitrarily close to $\mathcal{S}^{\mathbb{H}}_1$, hence the left hand side of the inequality can be arbitrarily small. On the other hand, the right hand side has no reason to be small, because the H-component of $\mathcal{F}^{n-1}W$ containing $\mathcal{F}^{n-1}x$ does not necessarily terminate near $\mathcal{F}^{n-1}x$.

2. Our proposed correction

We propose a variation on the definition of the function r_n which satisfies [2, Exercise 5.69], and then prove a growth lemma for this modified function.

2.1. **Finite horizon billiards.** It is instructive to first deal with finite horizon billiards, since this case requires a modification simpler than for the infinite horizon case, which will be considered in the next subsection. Let W be a weakly homogeneous unstable curve and, for any $x \in W$, denote with $W_n(x)$ the H-component of $\mathcal{F}^n W$ containing the point $\mathcal{F}^n x$. For $n \geq 0$ we let $\tilde{r}_n(x)$ denote the distance, measured along $W_n(x)$, from the point $\mathcal{F}^n x$ to $\partial W_n(x) \cup \mathcal{S}_1$. Observe that $\tilde{r}_n \leq r_n$. Then we propose the following correction of [2, Exercise 5.69].

Lemma 2.1. *There exists a constant $C > 0$ such that*

$$CE_s(\mathcal{F}^n x) d^s(\mathcal{F}^n x, \mathcal{S}_{-1}^{\mathbb{H}}) \geq \min\{\tilde{r}_{n-1}(x), E_s(\mathcal{F}^n x) r_n(x)\}.$$

Proof. First recall (see [2, (5.15)]) that

$$\mathcal{S}_{-1}^{\mathbb{H}} = \mathbb{S} \cup \mathcal{F}\mathbb{S} \cup \mathcal{S}_{-1},$$

hence for any $x \in \mathcal{M} \setminus \mathcal{S}_n$

$$d^s(\mathcal{F}^n x, \mathcal{S}_{-1}^{\mathbb{H}}) = \min\{d^s(\mathcal{F}^n x, \mathbb{S}), d^s(\mathcal{F}^n x, \mathcal{F}\mathbb{S}), d^s(\mathcal{F}^n x, \mathcal{S}_{-1})\}.$$

We proceed to estimate separately each of the three quantities on the right hand side. We begin with $d^s(\mathcal{F}^n x, \mathbb{S})$; recall that $\mathbb{S}$ is a countable union of horizontal curves. By definition it holds that $d^s(\mathcal{F}^n x, \mathbb{S}) \geq d(\mathcal{F}^n x, \mathbb{S})$; moreover since the unstable cone is uniformly transversal to the horizontal direction (see [2, (4.13)]) and since weakly homogeneous unstable curves cannot cross the boundaries $\mathbb{S}$ of homogeneity strips, we conclude that there exists $c > 0$ so that

$$d^s(\mathcal{F}^n x, \mathbb{S}) \geq c r_n(x).$$

Let us then consider the term $d^s(\mathcal{F}^n x, \mathcal{F}\mathbb{S})$. There are two possibilities: recall that $d^s(y, S) = \inf_{z \in S} d^s(y, z)$; either the $\inf$ is attained, or it is not. If it is attained, then[1], observe that the preimage of any stable curve linking $\mathcal{F}^n x$ to $\mathcal{F}\mathbb{S}$ is a stable curve linking $\mathcal{F}^{n-1} x$ to $\mathbb{S}$, whose length is expanded (see [2, (5.57)]) by $E_s(\mathcal{F}^n x)$; hence for some $c > 0$,

$$E_s(\mathcal{F}^n x) d^s(\mathcal{F}^n x, \mathcal{F}\mathbb{S}) \geq c d^s(\mathcal{F}^{n-1} x, \mathbb{S})$$

[1]This is, essentially the argument hinted in the book to solve Exercise 5.69, but bounds for the other two possibilities are missing.

and arguing (by uniform transversality) as in the previous case we conclude that

$$E_s(\mathcal{F}^n x) d^s(\mathcal{F}^n x, \mathcal{F}\mathbb{S}) \geq c r_{n-1}(x).$$

If, on the other hand, the $\inf$ is not attained, it means that it is attained for a limit point in $\mathrm{cl}\,(\mathcal{F}\mathbb{S}) \setminus \mathcal{F}\mathbb{S}$. However, the discussion of [2, page 110, see also Figure 5.2], implies that all accumulation points of $\mathcal{F}\mathbb{S}$ lie on $\mathcal{S}_{-1}$, which yields that in this case $d^s(\mathcal{F}^n x, \mathcal{F}\mathbb{S}) \geq d^s(\mathcal{F}^n x, \mathcal{S}_{-1})$, which we now proceed to treat.

Observe that any stable curve linking $\mathcal{F}^n x$ to $\mathcal{S}_{-1}$ is mapped by $\mathcal{F}^{-1}$ to a stable curve linking $\mathcal{F}^{n-1} x$ to $\mathcal{S}_1$; we thus obtain, for some $c > 0$

$$E_s(\mathcal{F}^n x) d^s(\mathcal{F}^n x, \mathcal{S}_{-1}) \geq c d^s(\mathcal{F}^{n-1} x, \mathcal{S}_1).$$

Once again observe that we have $d^s(\mathcal{F}^{n-1} x, \mathcal{S}_1) \geq d(\mathcal{F}^{n-1} x, \mathcal{S}_1)$. Moreover, any unstable curve W is uniformly transversal to any curve in $\mathcal{S}_1$ (see [2, (4.13) and (4.21)]). We conclude that we can find $c' > 0$ so that $d^s(\mathcal{F}^{n-1} x, \mathcal{S}_1) \geq c' \tilde{r}_{n-1}(x)$.

Our lemma then follows from grouping these estimates together, and observing that $\tilde{r}_{n-1} \leq r_{n-1}$. □

2.2. **Infinite horizon billiards.** In order to deal with the case of infinite horizon billiards, we will need an improvement on the bound given in Lemma 2.1. In order to state this improved bound, we find convenient to recall some observations from [2]. For ease of exposition, we restrict our analysis to infinite horizon billiards of the type described in [2] (see [2, Figure 2.4] for an example). Namely, we only consider periodic arrangements of scatterers which arise as unfoldings of some domain $\mathcal{D}$ on Tor^2. For such billiards, the following properties hold:

Proposition 2.2.

(a) *The singularity set $\mathcal{S}_1$ consists of countably many smooth compact curves (see* [2, Proposition 4.45]*);*
(b) *there exist finitely many accumulation points $\{x_1, x_2, \cdots x_s\}$ for those curves (see* [2, Exercise 4.51]*); all such points lie on $\mathcal{S}_0$;*
(c) *for any $k \in \{1, \cdots, s\}$, the singularity set in a neighborhood of x_k is described by* [2, Figure 4.15]

In particular, let $x_k \in \mathcal{S}_0$ denote one such accumulation point; then a sufficiently small neighborhood of x_k is intersected by countably many stable curves, which form a so-called *infinite horizon sequence.* Each curve in this sequence corresponds to a grazing collision with some scatterer that belongs to a fundamental domain at a distance at least[2] $\nu > 0$. Let us denote such curve with $\mathcal{S}_{1,k,\nu}$. This implies (see [2, (4.21)]) that there exists $c > 0$ so that $\mathcal{S}_{1,k,\nu}$ is tangent to the following cone:

$$\hat{\mathcal{C}}^s_\nu = \{-\mathcal{K} - c\nu^{-3/2} \le \frac{d\varphi}{dr} \le -\mathcal{K}\}; \tag{2.1}$$

in fact $\tau \sim \nu$ and [2, Exercise 4.52] implies that we have that $\cos\varphi \le c\nu^{-1/2}$ for some $c > 0$. Similar considerations imply that any stable curve in the cell $\mathcal{D}^+_{k,\nu}$ has to be tangent to the same cone.

Denote with $\mathcal{S}_1^*$ the union of those singularity curves in $\mathcal{S}_1$ that do not belong to any infinite horizon sequence, and let

$$\mathcal{S}_1^N = \mathcal{S}_1^* \cup \bigcup_k \bigcup_{\nu \le N} \mathcal{S}_{1,k,\nu}.$$

Then fix some $\bar{N}$ sufficiently large; since all curves in $\mathcal{S}_1^*$ do not belong to any infinite horizon sequence, it is possible to choose $\bar{N}$ so large that any curve in $\mathcal{S}_1^*$ will correspond to grazing collision with flight time at most[3] $\bar{N}$. For any weakly homogeneous unstable curve W and any $x \in W$, we thus define:

$$\tilde{r}_{W,*}(x) = \inf_{N > \bar{N}} \{N^{3/2} d_W(x, \mathcal{S}_1^N)\}.$$

where $d_W(x,\mathcal{S})$ denotes the distance of x from $\mathcal{S}$ measured along W, which can be defined as follows: let W' be the connected component of $W \setminus \mathcal{S}$ containing x; then x in turn cuts W' into two subcurves; we denote with $d_W(x,\mathcal{S})$ the length of the shortest of such subcurves. In particular $d_W(x,\mathcal{S}) = 0$ if $x \in \partial W$ or $x \in \mathcal{S}$. Finally, for $n \ge 0$, we introduce the shorthand notation

$$\tilde{r}_{n,*}(x) = \tilde{r}_{W_n(x),*}(x),$$

where $W_n(x)$ is the H-component of $\mathcal{F}^n W$ containing $\mathcal{F}^n x$.

Lemma 2.3. *There exists $K = K(\mathcal{D}) > 0$ depending only on the geometry of the billiard and $\delta > 0$ so that for any N sufficiently large, any unstable curve W with $|W| < \delta$ can intersect at most $K \cdot N$ singularity curves in $\mathcal{S}_1^N$.*

Proof (see also [2, Page 129]*).* Recall that since we are considering billiards without corner points, there exists $\tau_{\min} > 0$ so that between any two consecutive collisions we must have a flight time of at least $\tau_{\min}$. On the other hand, singularities curves in $\mathcal{S}_1^N$ have a flight time that is bounded above

[2]In [2], the symbol n is used instead of ν, but this will not work that well with our notation

[3]The definition of “infinite horizon sequence” in fact allows for some flexibility in the choice of such an $\bar{N}$.

by N; we conclude that at most $N/\tau_{\min}$ singularity curves in $\mathcal{S}_1^N$ may join at any point $x \in \mathcal{S}_1$; setting $K = 1/\tau_{\min}$ we conclude that any sufficiently short unstable curve can intersect at most $k \cdot N$ singularity curves in $\mathcal{S}_1^N$. □

We now show that the modified function also satisfies the statement of Lemma 2.1; namely:

Lemma 2.4. *There exists a constant* $C > 0$ *such that*

$$CE_s(\mathcal{F}^n x) d^s(\mathcal{F}^n x, \mathcal{S}_{-1}^{\mathbb{H}}) \geq \min\{\tilde{r}_{n-1,*}(x), E_s(\mathcal{F}^n x) r_n(x)\}.$$

Proof. The proof of the above lemma follows the same argument of the proof of Lemma 2.1; the modification is needed in the estimate of the last term. We proceed thus to estimate $d^s(\mathcal{F}^{n-1}, \mathcal{S}_1)$. Now we further decompose

$$\mathcal{S}_1 = \bigcup_{N > \bar{N}} \mathcal{S}_1^N,$$

hence

$$d^s(\mathcal{F}^{n-1}x, \mathcal{S}_1) = \inf_{N > \bar{N}} d^s(\mathcal{F}^{n-1}x, \mathcal{S}_1^N).$$

We now show that there exists C so that for any $N > \bar{N}$;

$$d^s(\mathcal{F}^{n-1}x, \mathcal{S}_1^N) \geq C \min_{\bar{N} \leq M \leq N} \{M^{3/2} d(\mathcal{F}^{n-1}x, \mathcal{S}_1^M)\} \tag{2.2}$$

This suffices to conclude, since unstable manifolds are uniformly transversal to stable manifolds and thus for some $c > 0$:

$$d(\mathcal{F}^{n-1}x, \mathcal{S}_1^M) \geq c \cdot d_{W_{n-1}(x)}(\mathcal{F}^{n-1}x, \mathcal{S}_1^M).$$

We prove the estimate (2.2) by induction on N; if $N = \bar{N}$, the estimate is trivial provided that $C \leq \bar{N}^{-3/2}$. Assume that (2.2) holds for $N - 1$, then

$$d^s(\mathcal{F}^{n-1}x, \mathcal{S}_1^N) = \min\{d^s(\mathcal{F}^{n-1}x, \mathcal{S}_1^{N-1}), d^s(\mathcal{F}^{n-1}x, \mathcal{S}_1^N \setminus \mathcal{S}_1^{N-1})\}$$

and by inductive hypothesis:

$$d^s(\mathcal{F}^{n-1}x, \mathcal{S}_1^N) \geq C \min\Big\{ \min_{\bar{N} \leq M \leq N-1} \{M^{3/2} d(\mathcal{F}^{n-1}x, \mathcal{S}_1^{N-1})\}, \\ d^s(\mathcal{F}^{n-1}x, \mathcal{S}_1^N \setminus \mathcal{S}_1^{N-1})\Big\}.$$

It thus suffices to show:

$$d^s(\mathcal{F}^{n-1}x, \mathcal{S}_1^N \setminus \mathcal{S}_1^{N-1}) \geq C N^{3/2} d(\mathcal{F}^{n-1}x, \mathcal{S}_1^N). \tag{2.3}$$

The proof of (2.3) follows from the observation that if a stable curve links some point y with $\mathcal{S}_1^N \setminus \mathcal{S}_1^{N-1}$, then necessarily $y \in \mathcal{D}_{k,\nu}$ for some k and some $\nu > N - 1$. By construction the whole stable curve has to belong to such cell, and therefore the angle between the stable curve and the singularity curve is bounded above by $cN^{-3/2}$ by (2.1). This implies (2.3). □

2.3. **Growth Lemma.** In this subsection we prove that the First Growth Lemma [2, Theorem 5.52] holds for the modified function $\tilde{r}_{n,*}$. We use the convention that $C_\#$ denotes some arbitrary positive constant whose value might be different from instance to instance.

Lemma 2.5 (Modified First Growth Lemma). *Let $W \subset \mathcal{M}$ be a sufficiently short unstable curve. There are constants $\hat{\Lambda} > 1, \vartheta_1 \in (0,1)$, and $c_1, c_2 > 0$ such that for all $n \geq 0$ and $\varepsilon > 0$:*

$$\mathbf{m}_W(\tilde{r}_{n,*}(x) < \varepsilon) \leq c_1(\vartheta_1\hat{\Lambda})^n \mathbf{m}_W(r_0 < \varepsilon/\hat{\Lambda}^n) + c_2\varepsilon\mathbf{m}_W(W). \tag{2.4}$$

By combining the above lemmata according to the strategy outlined in [2, page 137], we obtain a correct proof of [2, Theorem 5.67].

Before proving the Growth Lemma, we recall a useful fact for the reader's convenience:

Lemma 2.6 (Distortion bound, see [2, Lemma 5.27]). *There exists $C_{\mathrm{d}} > 0$ so that for any weakly homogeneous unstable manifold W, W_n a H-component of $\mathcal{F}^n W$ and $V_n = \mathcal{F}^{-n} W_n \subset W$, the following holds. For any measurable set E:*

$$C_{\mathrm{d}}^{-1} \frac{\mathbf{m}_{W_n}(E)}{\mathbf{m}_{W_n}(W_n)} \leq \frac{\mathbf{m}_{V_n}(\mathcal{F}^{-n}E)}{\mathbf{m}_{V_n}(V_n)} \leq C_{\mathrm{d}} \frac{\mathbf{m}_{W_n}(E)}{\mathbf{m}_{W_n}(W_n)}.$$

We now proceed with proving the Modified Growth Lemma; the main observation is the following

Lemma 2.7. *There exists $C > 0$ and $\delta > 0$ so that for any $|W| < \delta$*

$$\mathbf{m}_W(\tilde{r}_{W,*}(x) < \varepsilon) \leq \mathbf{m}_W(r_W(x) < C\varepsilon)$$

Proof. Let $\delta > 0$ be the one found in Lemma 2.3; then for any W with $|W| < \delta$ the set $\{\tilde{r}_{W,*}(x) < \varepsilon\}$ is a union of

- 2 intervals of size ε at the boundary of W;
- at most $K \cdot \bar{N}$ intervals of size $2C_\#\varepsilon$ centered at each point of $W_i \cap \mathcal{S}_1^{\bar{N}}$ (recall Lemma 2.3)
- at most one sequence of intervals $\{I_n\}_{n=\bar{N}}^{\infty}$ each of size $C_\# n^{-3/2}\varepsilon$, corresponding to a possible infinite horizon sequence.

Hence:

$$\mathbf{m}_W(\tilde{r}_{W,*}(x) < \varepsilon) \leq 2\varepsilon \left(1 + C_\# K\bar{N} + C_\# \sum_{n \geq \bar{N}} n^{-3/2}\right) < 2C\varepsilon, \tag{2.5}$$

that proves the lemma, since $\mathbf{m}_W(r_W(x) < C\varepsilon) \leq \min\{2C\varepsilon, \mathbf{m}_W(W)\}$. □

Proof of Lemma 2.5. In order to refer directly to some formulas in [2], we find convenient to introduce the *adapted length* $|\cdot|_*$, which is the length measured with respect to the adapted metric $\|\cdot\|_*$ defined in [2, page 127]. Let $\mathbf{m}^*_W$ denote the measure with respect to $|\cdot|_*$. Since the metric $\|\cdot\|_*$ and the Euclidean metric are uniformly equivalent (see [2, Exercise 5.54]), it suffices to show the following inequality

$$\mathbf{m}^*_W(\tilde{r}_{n,*}(x) < \varepsilon) \leq c_1(\vartheta_1\hat{\Lambda})^n \mathbf{m}^*_W(r_0 < \varepsilon/\hat{\Lambda}^n) + c_2\varepsilon\mathbf{m}^*_W(W). \tag{2.6}$$

Let us fix $\delta > 0$ sufficiently small; we now recall the definition of *shortened H-components*, which was introduced in [2, Theorem 5.52]. Given W, let W_1 be an arbitrary H-component of $\mathcal{F}W$; it might happen that $|W_1|_* > \delta$. If this is the case, we partition it in k curves of equal $|\cdot|_*$-length, where $k = \lfloor |W|_*/\delta \rfloor + 1$. The collection of all subcurves obtained by partitioning all H-components of $\mathcal{F}W$ constitutes is the set of *shortened H-components* of $\mathcal{F}W$ and will be denoted by $\{W'_{1,i}\}$. Inducing this construction on each shortened H-component of $\mathcal{F}^nW$ yields the set of *shortened H-components* of $\mathcal{F}^{n+1}W$ for $n \geq 1$. This construction leads naturally to the definition of $W'_n(x)$ to be that shortened H-component of $\mathcal{F}^nW$ which contains $\mathcal{F}^nx$ and of the corresponding function $r'_n(x) = r_{W_n(x)}(\mathcal{F}^nx)$.

Recall that the original Growth Lemma (or rather, its proof, see [2, Theorem 5.52]) implies that, for some $c'_1, c'_2 > 0$:

$$\mathbf{m}^*_W(r'_n < \varepsilon) \leq c'_1(\vartheta_1\hat{\Lambda})^n \mathbf{m}^*_W(r_0 < \varepsilon/\hat{\Lambda}^n) + c'_2\varepsilon\mathbf{m}^*_W(W). \tag{2.7}$$

Let W_n be a shortened H-component of $\mathcal{F}^nW$; then Lemma 2.7 implies:

$$\mathbf{m}^*_{W_n}(\tilde{r}_{W_n,*}(x) < \varepsilon) \leq \mathbf{m}^*_{W_n}(r_{W_n}(x) < C\varepsilon).$$

Using the distortion estimate given in Lemma 2.6 we conclude that, letting $V_n = \mathcal{F}^{-n}W_n$:

$$\mathbf{m}^*_{V_n}(\tilde{r}_{W_n,*}(\mathcal{F}^nx) < \varepsilon) \leq C_{\mathrm{d}}^2\mathbf{m}^*_{V_n}(r_{W_n}(x) < C\varepsilon)$$

Summing over all V_n's we obtain $\mathbf{m}^*_W(\tilde{r}_{n,*}(x) < \varepsilon) \leq C_{\mathrm{d}}^2\mathbf{m}^*_W(r'_n(x) < C\varepsilon)$, and using (2.7) we conclude that

$$\mathbf{m}^*_W(\tilde{r}_{n,*}(x) < \varepsilon) \leq C_1(\vartheta_1\hat{\Lambda})^n \mathbf{m}^*_W(r^*_0 < \varepsilon/\hat{\Lambda}^n) + C_2\varepsilon\mathbf{m}^*_W(W). \qquad \square$$

References

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