<u>Livšic Th'm</u> : Let X be cpt mfold, ϕ_t be Anosov, transitive

If $f \in C^{\text{Hölder}}(X)$ s.t.

$$\forall \underset{\text{orbit } \gamma}{\phi\text{-periodic}} \qquad \int_\gamma f \, ds = 0 \qquad \text{then}$$

f is a coboundary i.e. $\exists \beta \in C^{\text{Hölder}}(X)$, diff'ble along ϕ s.t.

$$\frac{d}{dt} \beta(\phi_t(x)) \Big|_{t=0} = f(x).$$

<u>Recall</u> : ϕ is transitive if $\exists x_0 \in X$ s.t.

$$\overline{\{\phi_t(x_0)\}_{t \in \mathbb{R}}} = X$$

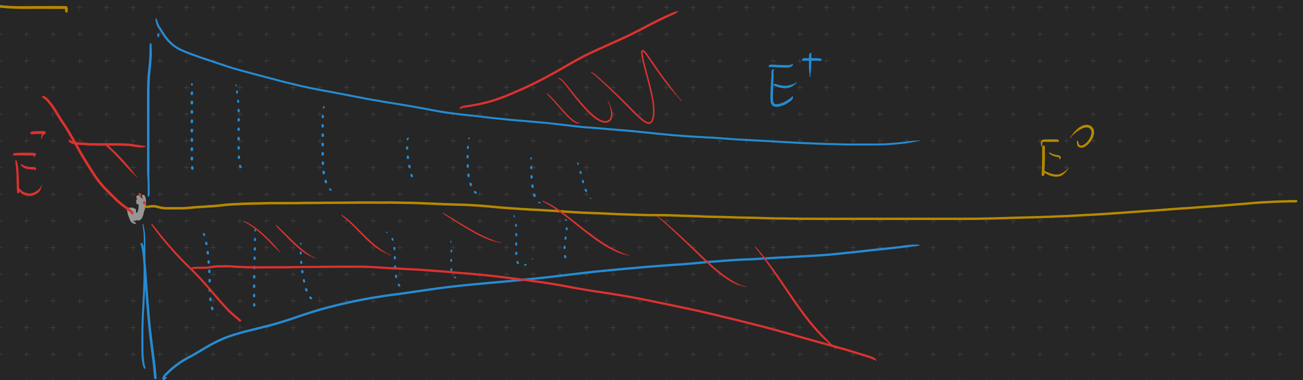
($\exists$ a dense orbit).

<u>Proposition</u> (*): Geodesic flow on (M, g) with negative curvature is

a) Anosov flow $\longleftarrow$

b) Transitive.

<u>Recall</u> : ϕ_t is Anosov if



Pf of Prop ⊛ a) 1. Geodesic flows are cone-hyperbolic

$\boxed{\text{2. Cone-hyp flows are Anosov.}}$

Def: $\mathbb{R}^n \supset C$ is a k-dim cone $(0<k<n)$ if it is closed, $\exists$ $\mathbb{R}^{n-k} \times \mathbb{R}^k \ni (x,y)$ coord. system, full rank $A: \mathbb{R}^k \to \mathbb{R}^n$ s.t.

$$C = \{ (x,y) \mid \|x\| \leq \|Ay\| \}$$

Example: A be the natural injection $y \mapsto (0, \varepsilon \cdot y)$

$a \in \mathbb{R}_{>0}$, the

Cone field is a map $x \longmapsto C_x \subset T_x M$

Def C, C' are transversal if $C \cap C' = \{0\}$

Def X Mfld ϕ_t smooth flow is cone-hyperbolic if there exist $\lambda < 1 < \mu$, a splitting

not nec'ly invariant

$$T_x M = E^0_x \oplus L^+_x \oplus L^-_x$$

and cone fields $C^\pm_x \subset T_x M$ s.t.

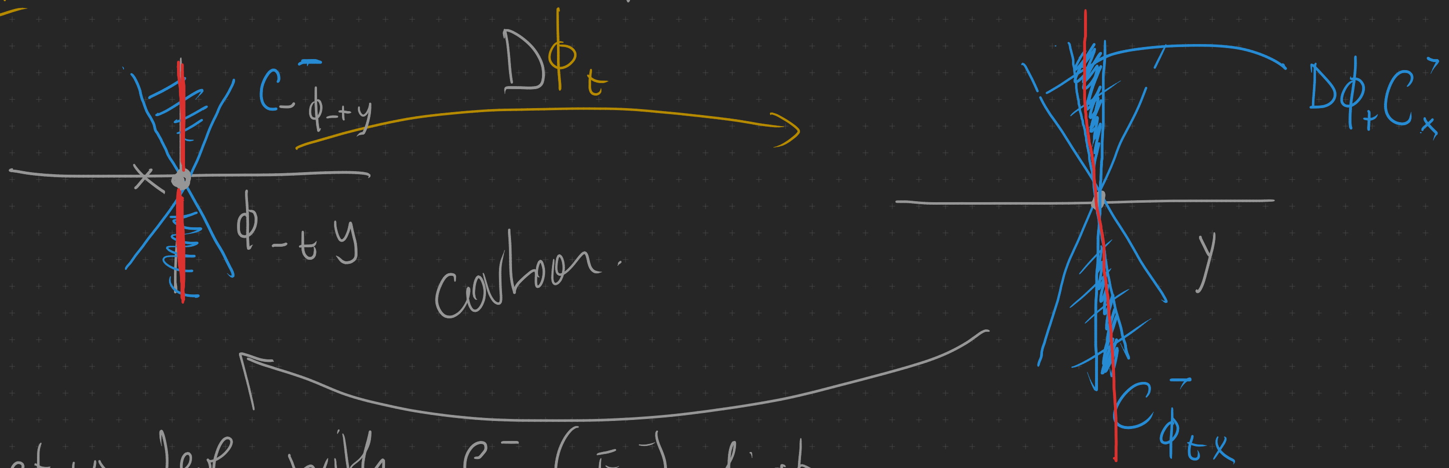
- E^0_x is the flow direction $\left(\text{i.e. } \frac{d}{dt} \phi_t(x)\big|_{t=0} \in E^0_x\right)$

- $L^{\pm}_x \subset C^{\pm}_x$. (this is required so that $e^{\pm}$ are transverse to each other + E^0_x)
- $D\phi_t \, \mathcal{C}^-_x \subset \operatorname{int} \mathcal{C}^-_{\phi_t x} \quad \forall t > 0$
- $D\phi_{-t} \, \mathcal{C}^+_x \subset \operatorname{int} \mathcal{C}^+_{\phi_{-t} x} \quad \forall t > 0$
- $\frac{d}{dt} \| D\phi_t \xi \| \Big|_{t=0} \geq \|\xi\| \log \mathcal{N} \quad \forall \xi \in \mathcal{C}^-_x$
- $\frac{d}{dt} \| D\phi_{-t} \eta \| \Big|_{t=0} \geq \|\eta\| \, |\log \lambda| \quad \forall \eta \in C^+_x$

Observe: Anosov flows are cone-hyperbolic. ✓

Prp: Cone-hyperbolic flows are Anosov.

Pf NT find invariant splitting of $T_x M = E^0 \oplus E^+ \oplus E^-$



Let us deal with $\mathcal{C}^-$ (E^-) first.

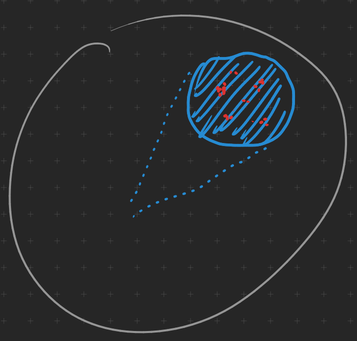
Fix $y \in X$. let $S_{t,y} = D\phi_t \, L^-_{\phi_{-t}(y)}$

$T_{t,y} = D\phi_t \, \mathcal{C}^-_{\phi_{-t}(y)}$

Notice by cone invariance if $t' > t$ $\; T_{t',y} \subset T_{t,y}$

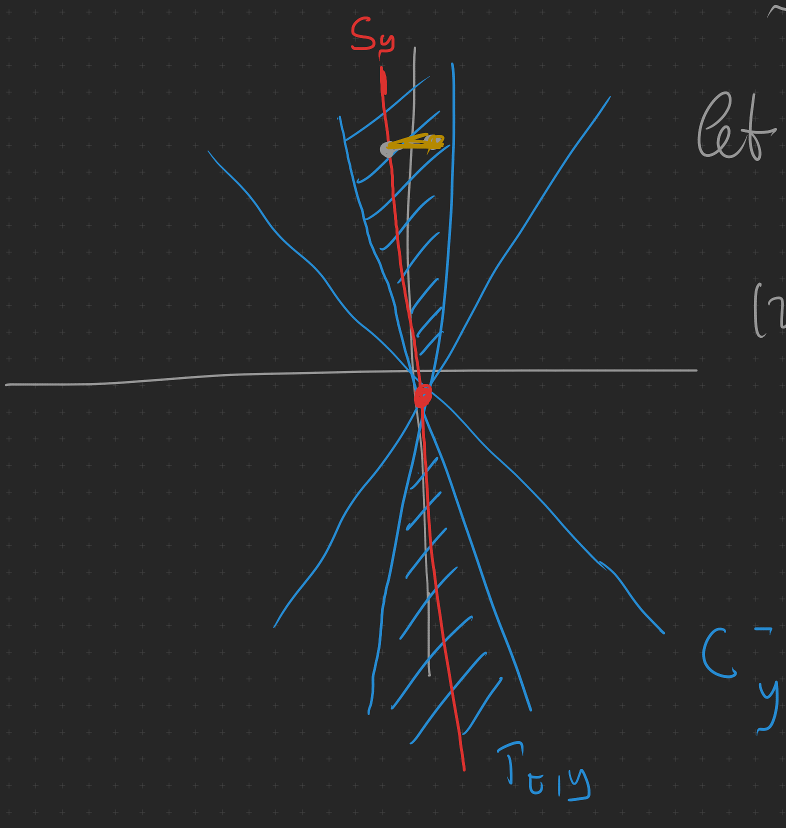
$$T_y = \bigcap_{n \in \mathbb{N}} T_{n,y} \neq \emptyset$$

observe $T_{t,y} \cap$ unit sphere of $T_y X$ is compact, Moreover, $\forall\, t' \geq t$ any $\{$orthonormal basis of $S_{t',y}\} \subset T_{t,y} \cap$ unit sphere



for $n \in \mathbb{N} \cap (t, \infty)$ we can consider an ordered orthonormal basis of $S_{n,y}$; extract a converging subsequence.

These converge to vectors u^i_y , span $u^i_y = S_y$



Let $(u, v) \in T_{t,y}$

$$\begin{array}{ccc} & \shortparallel & \\ (u, v') & + & (0, v'') \\ \Uparrow & & \Uparrow \\ S & & C^+_y \end{array}$$

$$\| D\phi_{-t}(u,v)\| \le \nu^{-t} \|(u,v)\|$$ by contraction in <0 ... of C^-

$$\| D\phi_{-t}(u,v')\| \le \nu^{-t} \|(u,v')\|$$

$$\| D\phi_{-t}(0,v'')\| \ge \lambda^{-t} \|v''\|$$ since $0, v'' \in C^+$

$$\|v''\| \le \lambda^{-1} D\phi_{-t}(0, v'')$$
$$\le \lambda^{t} \left(\| D\phi_{-t}(u,v)\| + \| D\phi_{-t}(u,v')\| \right)$$
$$\le \left(\frac{\lambda}{\nu}\right)^{t} \max\left(\|u,v\|, \|u,v'\|\right) \quad \forall t$$

$(u,v) \in S$

$$\Leftarrow \|v''\| = 0 \Longleftarrow (u,v) \in T_{t,y} \ \forall t > 0$$

$S \rightsquigarrow E^-$. By the same token we get E^+.

$\square$

<u>Prove</u>: Geodesic flows in <0 are cone-hyp.

<u>Recall</u>: (M,g) $\quad \phi_t : T^1M \circlearrowleft$ $\quad X$

$x = (p,v) \quad \|v\| = 1$

$\exists$ unique geodesic γ_x s.t. $\gamma_x(0) = p$
$\dot\gamma_x(0) = v$

$$\phi_t(p,v) = (\gamma_x(t), \dot\gamma_x(t)).$$

w.s. $\|\dot\gamma_x(t)\| = 1$.

Need to find ~~$T\, T^1M = E^0 \oplus E^+ \oplus E^-$~~

invariant cones for ϕ_t on TT^1M.

Def Let γ be a geodesic for (M,g)

$Y(t)$ is **Jacobi field (for γ)** if

$$\ddot Y(t) + K(t)Y(t) = 0$$

where $\dot{}$ is $\frac{d}{dt}$ along geodesic γ

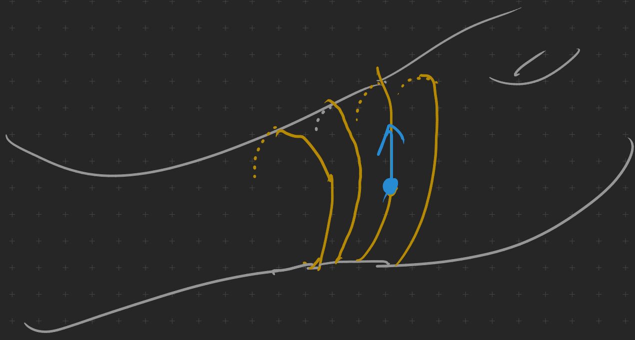
$$K(t) = R(\dot\gamma, \cdot)\dot\gamma$$

Fact: If (M,g) is <0 curvature, $K(t)$ is <u>negative</u> <u>definite</u> symmetric operator

$$\hookrightarrow \exists\, k > 0 \text{ s.t.}$$

$$\langle Kx, x\rangle < -k\langle x, x\rangle$$

Motivation if γ_s is a 1-parameter family of geodesics then



$\frac{d}{ds}\gamma_s(t) = Y(t)$

is a Jacobi field.

Jacobi fields describe the infinitesimal difference b/w a geodesics and an infinitely close geodesics.

Examples: • $\boxed{Y = \dot\gamma}$: $\ddot{Y} + R\,Y = 0$ ✓

$$\dddot\gamma + \underbrace{R\,\dot\gamma}_{0} = 0$$

$\gamma_s(t) = \gamma(t+s)$

• $\boxed{Y = t\dot\gamma}$ $\quad \underbrace{(t\dot\gamma)''}_{t\dddot\gamma + \ddot\gamma} + \underbrace{tR\,\dot\gamma}_{0} = 0$

$\gamma_s(t) = \gamma(s\cdot t)$ linear reparam. of γ

trivial Jacobi fields $\parallel$ $\dot\gamma$

In fact given any Y Jacobi field

$$Y = Y^{T} + Y^{\perp}$$

Y^{T}: tgt to $\dot\gamma$

$Y^{\perp}$: $\perp$ to $\dot\gamma$

E^0: translations along flow

~~linear rep. the flow.~~

Lemma: Let $x = (p, v)$ $\exists$ isomorphism

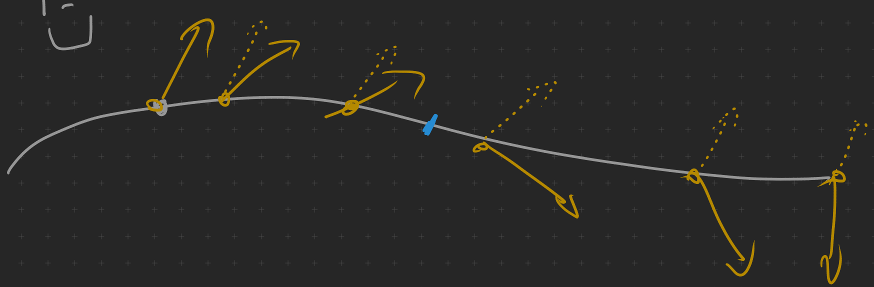
$$\Psi_x : T_x TM \to T_pM \oplus T_pM$$

$$\Xi = (x, w) \mapsto (Y, Y')$$

$$\Psi_{\phi_t x}(D\phi_t \Xi) = (Y(t), \dot Y(t))$$

where $\begin{cases} Y(0) = Y \\ \dot Y(0) = Y' \end{cases}$ Y is the Jacobi field solving $(*)$ + IC.

Ξ



by $(*)$ $\forall Y \perp \dot\gamma$ $\langle KY, Y\rangle \le -k\langle Y, Y\rangle$

$-k^2$ is a bound

Def Adapted norm on

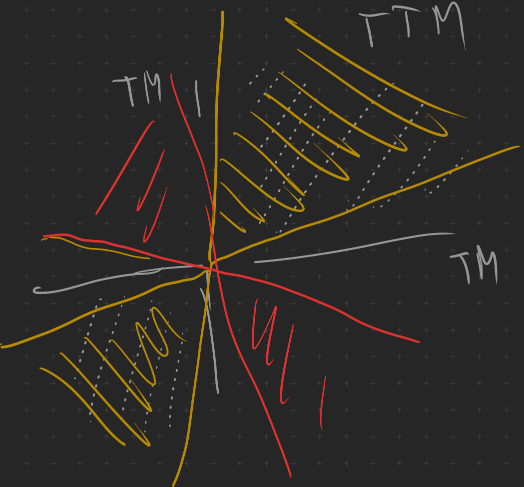
on the sect. curv of the Mfold.

twisted Sasaki metric.

$$T_pM \oplus T_pM$$

$$\|(u,v)\|_\varepsilon = \sqrt{\|u\|^2 + \varepsilon\|v\|^2}$$

Cone $C^-_{\delta,\varepsilon} = \left\{ (x,x') \in TTM \text{ s.t. } \dfrac{\langle x,x'\rangle}{\|(x,x')\|^2_\varepsilon} \geq \delta \right\}$



Prp: $\exists\, \delta,\varepsilon > 0$ s.t.

$D\phi_t C_{\delta,\varepsilon} \subset \operatorname{int} C_{\delta,\varepsilon}$

+ expansion in $C_{\delta,\varepsilon}$.

pf (M being a surface)

Crit

$len = \int n_\gamma$