Complex Variables
Lecture Notes

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Preface

These are Lecture Notes for MAT 334 "Complex Variables" at Faculty of Arts and Science, University of Toronto. This is a junior class for all but Math Specialist students.

I was teaching it for several years and the last time it at Fall of 2020. This time the class was taught online due to COVID-19 pandemic and I made beamer slides for lectures. These slides were reformatted to a book format.

These Lecture Notes are addition rather than substitution for our standard textbook Complex Variables, 2nd Edition, by Stephen D. Fisher (referred as Textbook).

We cover Chapters 1–3 from Textbook, however some material removed and other material added, exposition is different.

Chapter 1

The Complex Plane

Introduction to MAT334

We start a class called "Complex Variables" but more precisely it should be called *Functions of a Complex Variable* and even more precisely *Functions of One Complex Variable*.

In late 17-th century I. Newton diskovered that the decomposition of functions of one real variable into power series (which we call today *Taylor series*) is a very powerful tool:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{1.1.1}$$

with

$$a_n = \frac{1}{n!} f^{(n)}(x_0). \tag{1.1.2}$$

Starting from I. Newton mathematicians began to use this tool extensively and for more than 100 years a function was something that is given by a power series.

They did not care where this series converges (if it converges at all, except for $x = x_0$), and if it converges to $f(x)$. Today we call these functions *analytic* or, more precisely *real analytic*.

There were some problems with this approach: many functions are not differentiable, even for infinitely differentiable functions series (1.1.1) may not converge (except at $x = x_0$), or converge not to $f(x)$. Also comparison of decompositions at different points was not very obvious. So, if a function $f(x)$ depends only on x, why we need to deal with x_0?

Also, if $f(x)$ is defined on the whole real line $\mathbb{R} = (-\infty, \infty)$ we cannot say where such series converges.

F.e. for $f(x) = e^x$, $\cos(x)$, $\sin(x)$ Taylor series at $x = 0$ converges on $\mathbb{R}$, but for $f(x) = \dfrac{1}{x^2+1}$ it converges only at $[-1, 1]$.

Note that if $f(x)$ is given by the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{1.1.1}$$

then one can plug instead of x a complex number z and get a function of the complex variable

$$f(z) = \sum_{n=0}^{\infty} a_n (z - x_0)^n \tag{1.1.3}$$

(assuming that it converges somewhere).

We will get an *complex analytic function* $f(z)$ which for real $z = x$ coincides with $f(x)$. We say that $f(z)$ is an *analytic continuation of $f(x)$ to the complex domain.*

However we are still talking about power series, and they depend on the point, where we decompose $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1.1.4}$$

(now z_0 could be a complex number as well).

So, we have

Goal #1. Characterize functions of the complex variable $z = x + iy$ which are analytic, so that they can be decomposed into series (1.1.4).

The theory of analytic functions of a complex variable became a workhorse of analysis with multiple applications to mathematics itself, physics, engineering and through many mathematical uses to many other fields.

Why "Complex variables" is a hard class

So far you know real valued and vector valued (in particular, complex valued) functions of one real variable (Calculus I), real valued and vector valued (in particular, complex valued) functions of several real variables (Calculus II). Now we have a theory of the functions of the complex variable. But such functions could be considered as functions of two real variables, x and y, right?

However only some of them could be decomposed into power series with respect to z and such functions have many properties which general functions of x, y do not posses.

So the theory of such functions is different from both a theory of functions of one real variable and a theory of functions of two real variables.

One needs to think in the framework of theory of functions of a complex variable! For example, while function $\sin(z)$ is uniquely defined as an analytic continuation of $\sin(x)$, they are very different. In particular, the latter is bounded, and the former is not.

Think differently!!
Think like a complex analyst!!

We start slowly, but do not relax.

1.1 Complex Numbers

1.1.1 Complex Numbers: Definition

Complex numbers are pairs of real numbers (x, y) written as $z = x + iy$. When x, y run a real line $\mathbb{R}$, corresponding complex number z runs a *complex plane* $\mathbb{C}$. We have a one-to-one correspondence between $\mathbb{C}$ and a real plain $\mathbb{R}^2$: $\mathbb{C} \ni z = x + yi \leftrightarrow \boldsymbol{z} = x\boldsymbol{i} + y\boldsymbol{j} \in \mathbb{R}^2$ (here $\boldsymbol{i}$ and $\boldsymbol{j}$ are coordinate vectors on the real plane).

Definition 1.1.1. x is called *a real part* of z and is denoted as $z = \operatorname{Re}(z)$ (sometimes denoted as $x = \Re(z)$) and y is called *an imaginary part* of z and is denoted as $z = \operatorname{Im}(z)$ (sometimes denoted as $x = \Im(z)$).

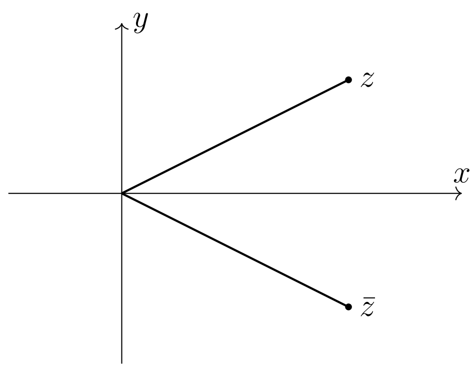
We identify $x + i0$ with a *real number* x and $0 + iy$ with an *imaginary number* $0 + iy$. In particular, we write $0 + i0$ simply as 0.

Therefore a complex plane $\mathbb{C}$ inherits from the the real plane $\mathbb{R}^2$ an addition $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$ and multiplication by a real number α: $\alpha(x + iy) = (\alpha x) + i(\alpha y)$. All the usual properties hold.

1.1.2 Complex Conjugate and Multiplication of Complex Numbers

There are two more operations:

Complex conjugate to $z = x + iy$ is $\bar{z} = x - iy$ (which is the reflection of z with respect to real line):



One can see easily that

$$
\begin{aligned}
&\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2,\\
&\overline{\alpha z} = \alpha \bar{z} \qquad \text{for real} \quad \alpha,\\
&\bar{\bar{z}} = z.
\end{aligned}
$$

Multiplication of complex numbers For complex numbers we introduce multiplication:

$$(x_1 + iy_1)(x_2 + iy_2) := (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \tag{1.1.5}$$

One can check easily that

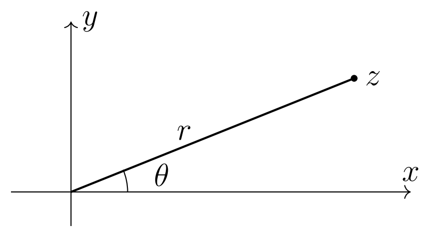
$$z_1z_2 = z_2z_1, \tag{1.1.6}$$
$$z(z_1 + z_2) = zz_1 + zz_2, \tag{1.1.7}$$
$$z_1(z_2z_3) = (z_1z_2)z_3, \tag{1.1.8}$$
$$\overline{z_1z_2} = \bar{z}_1\bar{z}_2. \tag{1.1.9}$$

Actually (1.1.5) is introduced to have these properties and

$$1 \cdot 1 = 1, \qquad i \cdot 1 = 1 \cdot i = i, \qquad i \cdot i = -1.$$

1.1.3 Polar Representation of Complex Numbers

Let us introduce *polar coordinates*: *polar radius* $r = \sqrt{x^2 + y^2}$ and *polar angle* θ:



- r is called *an absolute value (or module) of* z and denoted by $|z|$.
- θ is called *an argument of* z and denoted by $\arg(z)$.

Remark 1.1.1. (a) $\arg(z)$ is defined *only* as $z \neq 0$ ($\iff$ $r \neq 0$).

(b) $\arg(z)$ is defined modulo $2\pi n$, $n \in \mathbb{Z}$ and thus is a *multivalued function* of z.

(c) $\operatorname{Arg}(z)$ is a *single-valued function* of z; usually we define it so that $0 \leq \operatorname{Arg}(z) < 2\pi$.

Observe that

$$r^2 = |z|^2 = z\bar{z}, \tag{1.1.10}$$

$$|\bar{z}| = |z|, \qquad \arg(\bar{z}) = -\arg(z). \tag{1.1.11}$$

Therefore $z^{-1} := \frac{\bar{z}}{|z|^2}$ is an *inverse to* $z \neq 0$: $z^{-1}z = zz^{-1} = 1$; z^{-1} does not exist for $z = 0$.

Since $x = r\cos(\theta)$, $y = r\sin(\theta)$ we conclude that

$$z = r(\cos(\theta) + i\sin(\theta)) \tag{1.1.12}$$

with $r \geq 0$ defined uniquely and θ defined modulo $2\pi n$, $n \in \mathbb{Z}$. It is a *polar representation of complex number* z.

If $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$, then

$$\begin{aligned} z_1z_2 =& r_1r_2\bigl(\cos(\theta_1) + i\sin(\theta_1)\bigr)\bigl(\cos(\theta_2) + i\sin(\theta_2)\bigr) \\ =& r_1r_2\Bigl[\bigl(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\bigr) \\ & \qquad + i\bigl(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)\bigr)\Bigr] \\ =& r_1r_2\bigl[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\bigr] \end{aligned}$$

(remember trigonometric formulas!) and therefore

$$|z_1 z_2| = |z_1| \cdot |z_2| \tag{1.1.13}$$

and

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2). \tag{1.1.14}$$

Using the formula for inverse we get

Theorem 1.1.1 (De Moivre's Theorem)**.** *As* $z_2 \neq 0$

$$|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}, \tag{1.1.15}$$

$$\arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2). \tag{1.1.16}$$

Corollary 1.1.2. *As* $z \neq 0$, $n \in \mathbb{Z}$

$$|z^n| = |z|^n, \tag{1.1.17}$$

$$\arg(z^n) = n \arg(z). \tag{1.1.18}$$

1.1.4 Exponential Representation of Complex Numbers

Let us introduce the following complex function of the real variable $\theta \in \mathbb{R}$:

$$e^{i\theta} := \cos(\theta) + i \sin(\theta). \tag{1.1.19}$$

From the above it follows that

$$e^{i(\theta_1 + \theta_2)} = e^{i\theta_1} e^{i\theta_2}, \tag{1.1.20}$$

$$e^{\bar{i\theta}} = e^{-i\theta}. \tag{1.1.21}$$

Further,

$$(e^{i\theta})' = \big(\cos(\theta) + i \sin(\theta)\big)' = -\sin(\theta) + i\cos(\theta) = i\big(\cos(\theta) + i\sin(\theta)\big) \\ = ie^{i\theta}$$

Thus

$$(e^{i\theta})' = ie^{i\theta}. \tag{1.1.22}$$

Properties (1.1.20) and (1.1.22) show that this function behaves like an exponent, which justifies this definition.

Furthermore, if we define for $\lambda = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$,

$$e^{\lambda\theta} := e^{\alpha\theta} e^{i\beta\theta} = e^{\alpha\theta}\bigl(\cos(\beta\theta) + i\sin(\beta\theta)\bigr), \tag{1.1.23}$$

then

$$\bigl(e^{\lambda\theta}\bigr)' = \lambda e^{\lambda\theta} \tag{1.1.24}$$

which we used in ODE class.

It follows from the definition and properties of $e^{i\theta}$ that

$$\cos(\theta) = \frac{1}{2}\bigl(e^{i\theta} + e^{-i\theta}\bigr) = \operatorname{Re}(e^{i\theta}), \tag{1.1.25}$$

$$\sin(\theta) = \frac{1}{2i}\bigl(e^{i\theta} - e^{-i\theta}\bigr) = \operatorname{Im}(e^{i\theta}) \tag{1.1.26}$$

It follows from the trigonometric representation and (1.1.19)

$$z = re^{i\theta}, \qquad r = |z|, \quad \theta = \arg(z). \tag{1.1.27}$$

It is called *exponential representation of the complex number.*

1.1.5 Roots and Fractional Powers

Let us find all complex roots z of equation

$$z^n = w, \qquad w = \rho e^{i\varphi}. \tag{1.1.28}$$

We look at it also in the exponential form $z = re^{i\theta}$. Then

$$z^n = r^n e^{in\theta} = \rho e^{i\varphi}. \tag{1.1.29}$$

Therefore $r^n = \rho$ and $n\theta = \varphi + 2\pi m$, $m \in \mathbb{Z}$ and $r = \rho^{1/n}$, $\theta = \frac{1}{n}(\varphi + 2\pi m)$ and

$$z = w^{1/n} = |w|^{1/n} e^{\frac{1}{n}(\operatorname{Arg}(w) + 2\pi m)}, \qquad m \in \mathbb{Z}. \tag{1.1.30}$$

One can see easily that only for $m = 0, 1, \dots, n-1$ formula (1.1.30) gives different values, and then they repeat. Therefore, for $w \neq 0$ equation $z^n = w$ has n simple roots defined by (1.1.30) (for $w = 0$ it has just one root of multiplicity n).

Definition 1.1.2. (a) If p/q is an irreducible fraction (which means that $p, q \in \mathbb{Z}$, $q > 0$ do not have common divisors except ± 1), then we define q-valued function

$$z^{p/q} := |z|^{p/q} e^{i\frac{p}{q}(\operatorname{Arg}(z)+2m\pi)}, \qquad m = 0, 1, \ldots, q-1. \tag{1.1.31}$$

(b) If t is an irrational number, then we define ∞-valued function

$$z^t := |z|^t e^{it(\operatorname{Arg}(z)+2m\pi)}, \qquad m \in \mathbb{Z}. \tag{1.1.32}$$

(c) Moreover, the same is true if t is an arbitrary complex number (unless t is a real rational number).

Subsection 1.1.1* A Formal View of the Complex Numbers. Read it by yourself in the Textbook Stephen D. Fisher, Complex Variables (optional).

Remark (0ptional). (a) Complex numbers can be interpreted as 2×2 matrices

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = x\mathbf{1} + y\mathbf{i}, \qquad \mathbf{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{i} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and all operations on complex numbers correspond to the same operations on such matrices.

(b) We cannot extend algebra of complex numbers without losing some properties. There are more general "numbers" f.e. quaternions and octonians but their role in mathematics is rather minor.

1.2 Some Geometry

1.2.1 Norm and Inner Product

Let us explore the *geometry of the complex plane* $\mathbb{C}$. Recall that a complex number $z = x + iy \in \mathbb{C}$ matches to a vector $\boldsymbol{z} = x\boldsymbol{i} + y\boldsymbol{j}$ (later we will not distinguish them!)

One can see easily that

$$\boldsymbol{z_1} \cdot \boldsymbol{z}_2 = x_1 x_2 + y_1 y_2 = \mathrm{Re}(z_1 \bar{z}_2) = \mathrm{Re}(\bar{z}_1 z_2) \tag{1.2.1}$$

where $\boldsymbol{z}_1 \cdot \boldsymbol{z}_2$ is an the inner product of $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$. In particular

$$\|\boldsymbol{z}\| = \sqrt{\langle \boldsymbol{z}, \boldsymbol{z} \rangle} = \sqrt{\mathrm{Re}(z\bar{z}) = \sqrt{|z|^2}} = |z| \tag{1.2.2}$$

and

$$|z_1 + z_2| \le |z_1| + |z_2| \tag{1.2.3}$$

where inequality become an equality if and only if either $z_2 = \alpha z_1$ with $\alpha \ge 0$ (and therefore $\alpha \in \mathbb{R}$) or $z_1 = 0$. This is a *triangle inequality.*

1.2.2 Straight Lines

Therefore straight line ℓ in $\mathbb{C}$ is defined by equation

$$\mathrm{Re}(\bar{n} z) = C \qquad n = A + iB \ne 0. \tag{1.2.4}$$

Indeed, we know that straight line ℓ in $\mathbb{R}^2$ is defined by equation

$$\boldsymbol{n} \cdot \boldsymbol{z} = C \tag{1.2.5}$$

where $\boldsymbol{n} = A\boldsymbol{i} + B\boldsymbol{j}$ is a normal to ℓ and

$$\frac{C}{\sqrt{A^2 + B^2}} = d \tag{1.2.6}$$

is the distance (with the sign) from the origin to ℓ.

1.2.3 Circles

A *circle with a center at* $p \in \mathbb{C}$ *and radius* $r \ge 0$ is defined by equation

$$|z - p| = r. \tag{1.2.7}$$

We say “a circle” meaning “circumference” (a line). The figure inside will be called “*a disk*”.

Consider now two distinct points, $p, q \in \mathbb{C}$ and a curve which is the locus of the points, satisfying

$$|z - p| = \rho |z - q| \qquad 0 < \rho < \infty. \tag{1.2.8}$$

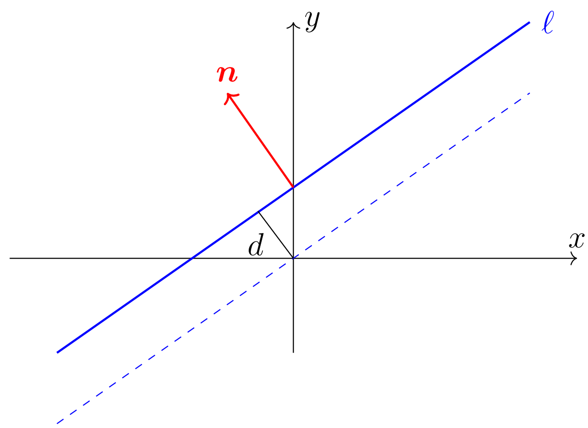


Figure 1.1: $d > 0$ above dashed line, $d < 0$ below dashed line.

If $\rho = 1$ we have points z equidistant from p and q. The locus of these points is precisely the straight line that is the perpendicular bisector of the line segment joining p to q.

However, if $\rho \neq 1$, the locus will be a circle. To see this, suppose that $0 < \rho < 1$ (otherwise, divide both sides of the equation by ρ). Let $z = w + q$ and $c = p - q$; then (1.2.8) becomes $|w - c| = \rho|w|$.

Upon squaring and transposing terms, this can be written as

$$(1-\rho^2)|w|^2 - 2\,\mathrm{Re}(\bar{c}w) + |c|^2 = 0.$$

Completing the square on the left side we find that

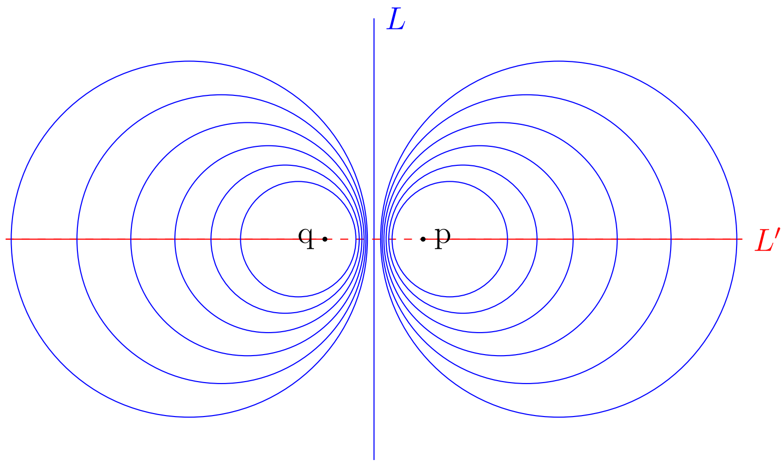
$$(1-\rho^2)\big|w - \frac{c}{1-\rho^2}\big|^2 = \frac{\rho^2|c|^2}{1-\rho^2}$$
$$\iff \big|w - \frac{c}{1-\rho^2}\big| = \frac{\rho|c|}{1-\rho^2};$$

Thus, w is on the circle of radius $R = \frac{\rho|c|}{1-\rho^2}$ centered at the point $w_0 = \frac{c}{1-\rho^2}$ and so z lies on the circle of the same radius R centered at the point

$$z_0 = \frac{p}{1-\rho^2} - \frac{\rho^2 q}{1-\rho^2}. \tag{1.2.9}$$

Point z_0 belongs to a straight line passing through p and q, on the distance $\frac{\rho^2|c|}{1-\rho^2}$ from p and further from q than p. Recall that $|z-p| = \rho|z-q|$

and $z_0 = \frac{p}{1-\rho^2} - \frac{\rho^2 q}{1-\rho^2}$. When $\rho \searrow 0$, circles became smaller and smaller and shrink to p, and when $\rho \nearrow 1$ then circles become larger and larger, approximating L and their centers go to the right infinity (in our picture), when ρ jumps over 1 we get circles on the other side, and their centers far away to the left, and when $\rho \nearrow \infty$ these circles shrink to q and their centers move to from the left to q.



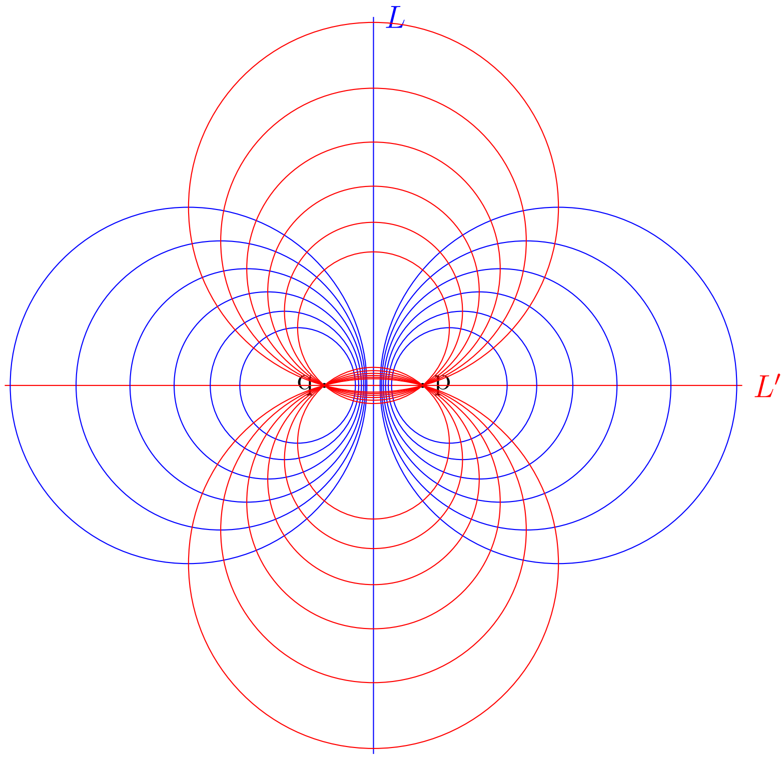
These are *Circles of Appolonius*.

The family of red circles (including straight line L') and the family of blue circles (including straight line L) are *mutually orthogonal*: each red circle is orthogonal to each blue circle at the point of intersection (two lines are orthogonal at the point of intersection if their tangent lines at this point are orthogonal).

See the proof on pp. 18–20 of the Textbook.

Remark 1.2.1. (a) In this class we will encounter many mutually orthogonal families of lines.

(b) We see that straight lines could be also included in the family of circles. We will see that a straight line is a circle, passing from the infinity (the exact meaning will be explained later).



1.2.4 Transformations: Translations, Scalings, Rotations, Inversion

Consider some important geometrical transformations.

Translations.

$$\mathbb{C} \ni z \mapsto w = z + a \in \mathbb{C}, \qquad a \in \mathbb{C}, \tag{1.2.10}$$

is a *translation by* a (also called *shift by* a). Obviously, it transforms straight lines and circles into straight lines and circles respectively.

Scalings.

$$\mathbb{C} \ni z \mapsto w = \lambda z \in \mathbb{C}, \qquad \lambda > 0, \tag{1.2.11}$$

is a *scaling by the factor* λ. Obviously, it transforms straight lines and circles into straight lines and circles respectively. If $0 < \lambda < 1$ it is a compression by the factor λ^{-1}.

Rotations.

$$\mathbb{C} \ni z \mapsto w = ze^{i\varphi} \in \mathbb{C}, \qquad \varphi \in \mathbb{R}, \tag{1.2.12}$$

is a *rotation by angle φ in the counter-clockwise direction about the origin*. Indeed, it preserves a module $|z|$ but adds φ to a polar angle $\arg(z)$. Obviously, it transforms straight lines and circles into straight lines and circles respectively.

If $\varphi < 0$ it is a rotation by angle $-\varphi$ in the clockwise direction about the origin.

In particular, $z \mapsto bz$ with $b \in \mathbb{C} \setminus \{0\}$ is a scaling by the factor $|b|$ and rotation by the angle $\arg(b)$.

Consider a more complicated map:

Inversion.

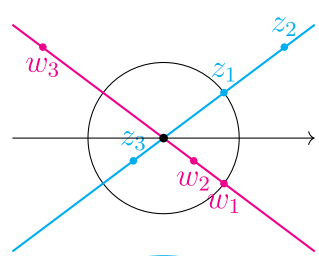
$$\mathbb{C} \setminus \{0\} \ni z \mapsto w = z^{-1} \in \mathbb{C} \setminus \{0\}; \tag{1.2.13}$$

it is called an *inversion*.

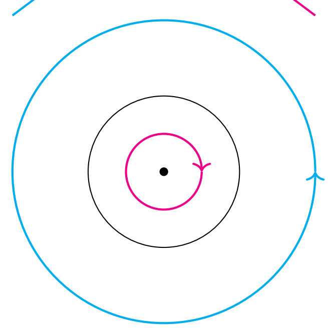
Later we will prove that it transforms circles and straight lines into circles and straight lines (but some straight lines can become circles and some circles can become straight lines–think, which are suspect!)

We can solve easier problems today:

Show that a straight line passing through 0 (with an angle φ) becomes the straight line passing through 0 (with an angle $-\varphi$).



Show that a circle about 0 (with a radius r) becomes the circle about 0 (with a radius r^{-1}) and direction changes (from counter-clockwise to clockwise).



Consider a circle passing through 0. We claim that it becomes a straight line, not passing through 0.

Let us make first a rotation $z \mapsto e^{i\varphi} z$ so after this we get a circle with a center on the positive half-line of $\mathbb{R}$ and passing through 0: $\{z\colon |z-r| = r\}$. But $w = z^{-1}$ after this also rotates albeit in the opposite direction: $w \mapsto e^{-i\varphi} z$.

Therefore

$$z = r(1+e^{it}) \implies w = \frac{1}{r(1+e^{it})} = \frac{(1+e^{-it})}{r(1+e^{it})(1+e^{-it})}$$
$$= \frac{(1+e^{-it})}{2r(1+\cos(t))} = \frac{(1+\cos(t) - i\sin(t))}{2r(1+\cos(t))} = \frac{1}{2r} - \frac{i\sin(t)}{2r(1+\cos(t))}$$

and $\operatorname{Re}(w) = \frac{1}{2r}$ and

$$\operatorname{Im}(w) = -\frac{\sin(t)}{2r(1+\cos(t))} = -\frac{1}{2r}\tan(t/2)$$

(check it!) and when t runs from 0 to $\pm\pi$, $\operatorname{Im}(w)$ runs from 0 to $\mp\infty$.

So, we got a vertical straight line $\{z\colon \operatorname{Re}(z) = 1/(2r)\}$.

Since *inversion is self-inverse* we conclude also that a straight line, not passing through 0 becomes a circle passing through 0.

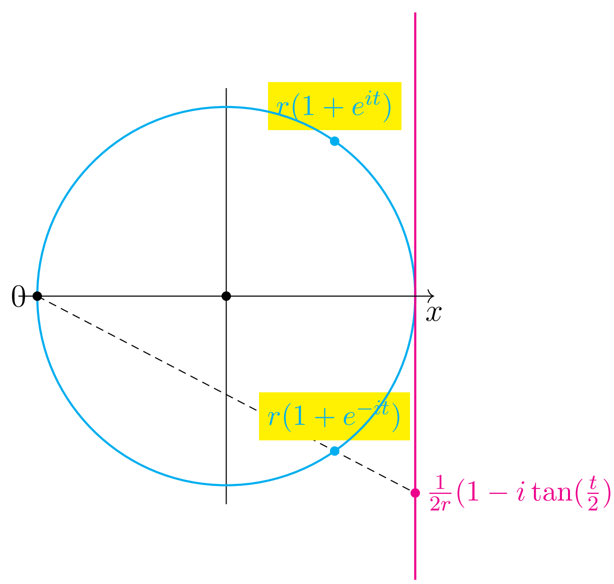


Figure 1.2: $r = \frac{1}{2}$

Example 1.2.1. Find the locus of points satisfying

$$\arg\left(\frac{z-p}{z-q}\right)=\varphi \tag{1.2.14}$$

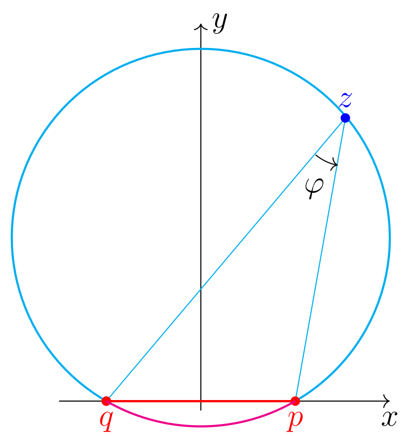
with fixed φ.

Solution. Note that (1.2.14) does not change under translations, rotations and even scalings. So without any loss of generality one can assume that $p=r$ and $q=-r$ (shifting their midpoint to 0 and then rotating). Assume that $0<\varphi<\pi$ (otherwise we permute p and q).

Also note that $\arg(z-p)$ is an angle between vector $\vec{pz}$ and the positive direction of axis x, and $\arg(z-q)$ is an angle between vector $\vec{qz}$ nd the positive direction of axis x. Therefore (1.2.14) means that

$$\varphi=\arg\left(\frac{z-p}{z-q}\right)= \\ \arg(z-p)-\arg(z-q)=\angle(qzp)$$

(in counter-clockwise direction) and from geometry it is known that the locus of such points is an arc.



Remark 1.2.2. The remaining arc of the circle is defined by $\arg\left(\frac{z-p}{z-q}\right)=\varphi-\pi$.

□

Remark 1.2.3 (optional). In Euclidean geometry there is also inversion transform $\boldsymbol{r}\mapsto\boldsymbol{r}|\boldsymbol{r}|^{-2}$ (in any dimension). But it is different. In complex variables it corresponds to $z\mapsto\bar{z}^{-1}$.

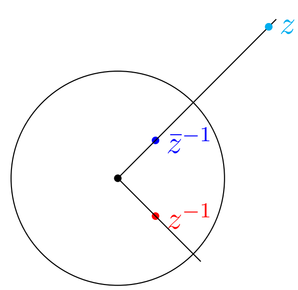


Figure 1.3: Complex variables inversion vs geometric inversion

1.3 Subsets of the Plane

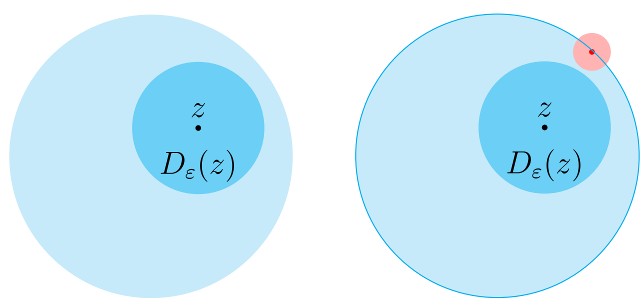
1.3.1 Neighbourhoods and Open Sets

We study today the *topology of complex plane* $\mathbb{C}$. Except the last part, covering *infinity* ∞ and topology of the *extended complex plane* $\widehat{\mathbb{C}}$, we already know it: topology on $\mathbb{C}$ coincides with topology on the ordinary plane $\mathbb{R}^2$.

A *neigbourhood* (or *vicinity*) of $z_0 \in \mathbb{C}$ is any *open disk* $D_\varepsilon(z_0) := \{z : |z - z_0| < \varepsilon\}$ of radius $\varepsilon > 0$ centered at z_0. Sometimes we say explicitly ε-*neigbourhood* (or ε-*vicinity*).

Definition 1.3.1. (a) *Interior of set* $M \subset \mathbb{C}$ is a set $\mathring{M}$ of points z, which are contained in M together with some vicinity $D_\varepsilon(z)$, ε depends on z.

(b) $M \subset \mathbb{C}$ is an *open* if $\mathring{M} = M$, which means that some neighbourhood every point $z \in M$ belongs to M.



(a) This set is open (b) This set is not

Remark 1.3.1. The notion of an open set depends where we consider it (and the same will be true for the notions of the interior of a set, a closed set, a closure of the set). For example Open interval $(a, b) \subset \mathbb{R}$ is an open subset of $\mathbb{R}$ but not of $\mathbb{C}$ and its interior in $\mathbb{C}$ is empty:

See other examples on page 23 of the Textbook.

1.3.2 The Boundary of a Set

Definition 1.3.2. (a) A point z is a boundary point of a set M if every open disk centered at z contains both points of M and points not in M.

(b) The set of all boundary points of a set M is called the *boundary of M and denoted ∂M*:

$$\partial M = \{z\colon \forall \varepsilon > 0 \; D_\varepsilon(z) \cap M \neq \emptyset,\; D_\varepsilon(z) \setminus M \neq \emptyset\}.$$

Theorem 1.3.1. *(i) Set M is open if and only if it contains no boundary point.*

(ii) Interior of the set M is a set of all points, belonging to M but not to its boundary.

(iii) The boundaries of M and of its complement $\mathbb{C} \setminus M = \{z\colon z \notin M\}$ coincide: $\partial M = \partial(\mathbb{C} \setminus M)$.

See examples on pages 24–25 of the Textbook.

1.3.3 Closed Sets

Closed sets

Definition 1.3.3. (a) A set M is closed if it contains its boundary: $M \supset \partial M$.

(b) A closure of the set M is a union of M and its boundary: $\bar{M} = M \cup \partial M$.

Theorem 1.3.2. *A set M is closed if and only if its complement $\mathbb{C} \setminus M = \{z\colon z \notin M\}$ is open.*

1.3.4 Connected Sets

Definition 1.3.4. (a) A *polygonal curve* is the union of a finite number of directed line segments P_1P_2, P_2P_3, …, $P_{n-1}P_n$ where the terminal point of one is the initial point of the next (except for the last one).

(b) *An open set M is connected* if each pair of $z \in M$, $w \in M$ may be joined by a polygonal curve lying entirely with M. That is, there are points $P_2, \dots, P_{n-1}$ such that all the line segments such that all line segments $P_1P_2, \dots, P_{n-1}P_n$ belong to M, $P_1 = z$, $P_n = w$.

(a) This set is connected

(b) This set is connected

(c) This set is not connected

(d) This set is not connected

See other examples on page 26 of Textbook.

Remark 1.3.2. We introduced a notion of connectivity *only for open sets*. Because of this connected sets remain the same if instead of polygonal curves in their definition we consider continuous curves, or smooth curves, etc.

Definition 1.3.5. An open connected set is called *a domain*.

Domains are the natural setting for the study of analytic and harmonic functions.

Definition 1.3.6. A set M is *convex* if the line segment zw joining each pair of points $z \in M$ and $w \in M$ also lies in M.

Definition 1.3.7. A set M is *starlike* if there is a point $z \in M$ such that line segment zw joining each pair of points z with any point $w \in M$ also lies in M.

Obviously, any convex open set is also starlike, and any starlike set is also connected.

(a) Convex set

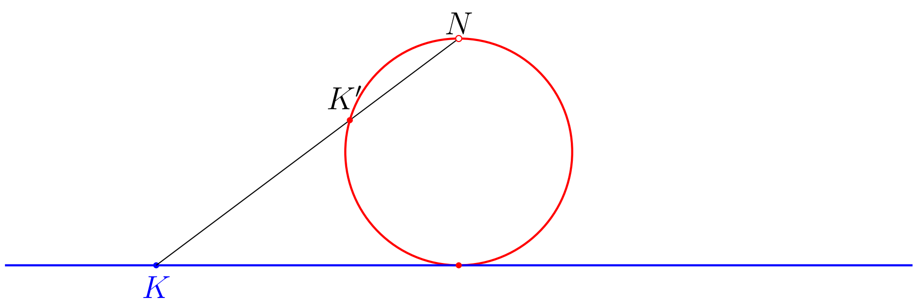
(b) Starlike but not convex set

(c) Connected but not starlike set

1.3.5 The Point at Infinity

Finally, let us introduce the point at infinity (∞) which will be treated mostly as an ordinary point.

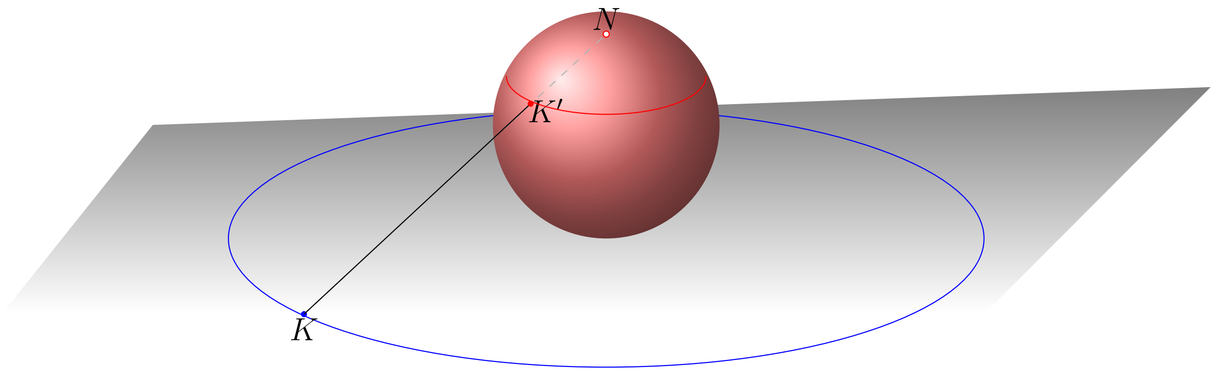
Let us start from the toy-model. Consider a circle $\mathbb{S}^1$, touching at 0 a straight line $\mathbb{R}$. The opposite point N will be a "North Pole".



For each point $K \in \mathbb{R}$ we draw a straight line KN; denote by K' it's intersection with $\mathbb{S}^1$. We get a bijection $\mathbb{R}$ and $\mathbb{S}^1 \setminus N$. We add to $\mathbb{R}$ a point ∞ which in this bijection corresponds to N. So we get a bijection between $\mathbb{S}^1$ and $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ which we call *extended real line. Note that in this construction there is just one infinity ∞, not two $+\infty$ and $-\infty$.*

Now let us consider a sphere $\mathbb{S}^2$, touching at 0 a plane $\mathbb{R}^2 \simeq \mathbb{C}$. The opposite point N will be a "North Pole".

For each point $K \in \mathbb{C}$ we draw a straight line KN; denote by K' it's intersection with $\mathbb{S}^2$. We get a bijection $\mathbb{C}$ and $\mathbb{S}^2 \setminus N$. We add to $\mathbb{C}$ a point ∞ which in this bijection corresponds to N. So we get a bijection between $\mathbb{S}^2$ and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which we call *extended complex plane.* Also a parallel of $\mathbb{S}^2$ passing through K' corresponds to a circle centered at 0 passing through K. Again, in this construction is *just one infinity* ∞.



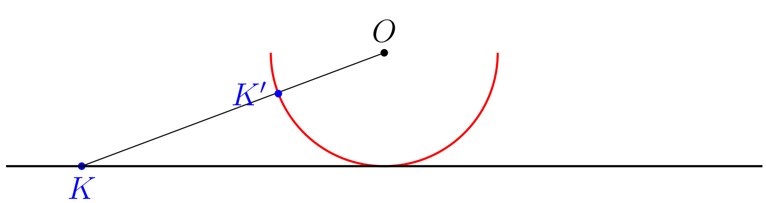
Note that in this correspondence to the neighbourhood of N on $\mathbb{S}^2$ (a cap above red parallel) corresponds the *exterior of the blue circle* while to the rest of $\mathbb{S}^2$ corresponds an interior of the blue circle $D_R(0)$.

Therefore, the *neighbourhood of* ∞ *is* $\{z\colon |z| > R\}$ *with arbitrarily large* R.

Recall the inversion $z \mapsto w = \frac{1}{z}$. Now we can say that it maps $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$, and 0 to ∞, and ε-neighbourhood of 0 onto ε^{-1}–neighbourhood of ∞, and ∞ to 0, and R-neighbourhood of ∞ onto R^{-1}–neighbourhood of 0.

In particular, *set* $M \subset \widehat{\mathbb{C}}$ *containing* ∞, *is open only if it contains neighbourhood of* ∞.

Remark 1.3.3 (optional). (a) Alternatively, $\mathbb{R}$ could be extended with two infinite points: as shown on this figure with $+\infty$ and $-\infty$ identified with left and right ends of semicircle:



(b) Similar multiple infinities can be introduced in extended real plane $\widehat{\mathbb{R}^2}$.

(c) But in complex variables we never do this: *in* $\widehat{\mathbb{C}}$ *there is just one infinite point.*

1.4 Functions and Limits

1.4.1 Definitions

Definition 1.4.1. (a) *A function of a complex variable* z is a map from domain $D \subset \mathbb{C}$ to $\mathbb{C}$: $f\colon D \to \mathbb{C}$, or more detaily, $D \ni z \to w = f(z) \in \mathbb{C}$. Reminder: domain is an open connected set.

(b) D is called *domain of definition of* f.

(c) The set of all possible values is called *a range of* f.

Remark 1.4.1. (a) If a function is given by some formula, but domain is not indicated, we consider *a natural domain*, where f is defined by this formula.

(b) We are not interested in all functions. Severe restrictions will be imposed late.

Example 1.4.1. (a) $f(z) = z^3 - 1$ has a domain $\mathbb{C}$, and the range $\mathbb{C}$ as well.

(b) $f(z) = \dfrac{1}{z-i}$ has a domain $\mathbb{C} \setminus \{i\}$ (all $\mathbb{C}$ except i) and a range $\mathbb{C} \setminus \{0\}$ (all $\mathbb{C}$ except 0).

(c) $f(z) = \bar{z}$ has a domain $\mathbb{C}$ and a range $\mathbb{C}$.

(d) $f(z) = \operatorname{Re}(z)$ has a domain $\mathbb{C}$ and a range $\mathbb{R}$.

(e) $f(z) = \dfrac{i}{\operatorname{Re}(z)}$ has a domain $\mathbb{C} \setminus i\mathbb{R}$ (all $\mathbb{C}$ except of the imaginary axis) and a range $i\mathbb{R} \setminus \{0\}$ (all imaginary axis exept 0).

Example 1.4.2. Function $f(z) = \dfrac{1+z}{1-z}$ with a domain (now domain is indicated!) $D = \{z\colon |z| < 1\}$ has a range $\{w\colon \operatorname{Re}(w) > 0\}$.

Indeed,

$$\operatorname{Re}\left(\frac{1+z}{1-z}\right) = \operatorname{Re}\left(\frac{(1+z)(1-\bar{z})}{(1-z)(1-\bar{z})}\right) = \frac{1-|z|^2}{|1-z|^2}$$

which is positive for $|z| < 1$.

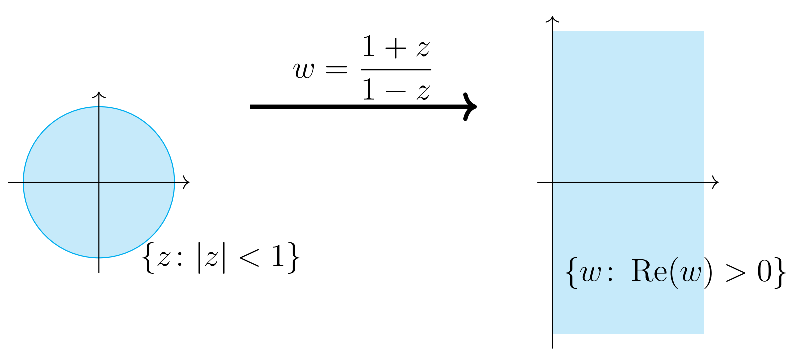
On the other hand, solving $w = \dfrac{1+z}{1-z}$ we get $z = \dfrac{w-1}{1+w}$. If $\operatorname{Re}(w) > 0$ then $|w-1| < |w+1|$ and therefore $|z| < 1$.

1.4.2 Graphs

We would like to plot functions of complex variables, but this is not really possible. Indeed, the plot of a real-valued function of a real variable is a two-dimensional picture.

The plot of a real-valued function of the complex variable (that is of two real variables) is a three-dimensional picture which could be plotted.

But the plot of a complex-valued function of a complex variable is four dimensional picture, so we draw domain and range. F.e. in Example 1.4.2 we show a picture like this:



Remark 1.4.2. Wolfram's Mathematica and Waterloo Maple provide visualization for complex functions using ComplexPlot but the result is colour-coded and really difficult to decipher.

1.4.3 Limits

The notions of limits and continuity is due to the same notions from Calculus I and Calculus II.

Definition 1.4.2. (a) A sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ *converges to* A (equivalently, *has a limit* A),

$$z_n \to A \ \text{ as } \ n \to \infty \qquad \text{or} \qquad \lim_{n\to\infty} z_n = A$$

if for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $n \geq N \implies |z_n - A| < \varepsilon$.

(b) Otherwise we say that $\{z_n\}_{n=1}^{\infty}$ *diverges.*

See examples on pages 33–34 of the Textbook.

Theorem 1.4.1 (Theorem 1 from page 34 of the Textbook)**.** *Let* $z_n \to A$*,* $w_n \to B$*,* $\alpha \in \mathbb{C}$*. Then*

(i) $\alpha z_n \to \alpha A$*,* $z_n + w_n \to A + B$*,* $z_n w_n \to AB$*;*

(ii) $\bar{z}_n \to \bar{A}$*,* $|z_n| \to |A|$*,* $\mathrm{Re}(z_n) \to \mathrm{Re}(A)$*,* $\mathrm{Im}(z_n) \to \mathrm{Im}(A)$*;*

(iii) If $B \neq 0$ *then* $\dfrac{z_n}{w_n} \to \dfrac{A}{B}$.

Proof. Proof is trivial. See on page 34 of the Textbook. □

Definition 1.4.3. Consider $f(z)$ in the domain D and let z_0 belong to the closure of M, that is is either in M or on its boundary. We say that $f(z)$ *tends to* L *as* z *tends to* z_0, or $f(z) \to L$ *as* $z \to z_0$, or $f(z)$ *has a limit* L *at* z_0, or

$$\lim_{z\to z_0} f(z) = L$$

if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that

$$z \in M,\ |z - z_0| < \delta \implies |f(z) - L| < \varepsilon.$$

See examples from page 35 of the Textbook.

Consider now a limit at ∞.

Definition 1.4.4. Assume that ∞ belongs to D, or at least, D is *unbounded* (domain is *bounded* if it is contained in some large disk: $D \subset \{z \colon |z| < R\}$ for some R, domain is *unbounded* if it is not contained in any disk: $D \cap \{z \colon |z| > R\} \neq \emptyset$ for any R.

We say that $f(z)$ *tends to* L *as* z *tends to* ∞, or $f(z) \to L$ *as* $z \to \infty$, or $f(z)$ *has a limit* L *at* ∞, or

$$\lim_{z\to\infty} f(z) = L$$

if for any $\varepsilon > 0$ there exists $R = R(\varepsilon)$, such that

$$z \in M,\ |z| > R \implies |f(z) - L| < \varepsilon.$$

See examples from pages 35–36 of the Textbook.

Example 1.4.3. Function $f(z) = \dfrac{z}{\bar{z}}$ has no limits as $z \to 0$ or $z \to \infty$.

Indeed, as $\arg(z) = \theta$, $f(z) = e^{2i\theta}$ so the limit exists along straight lines but depend on the direction.

Theorem 1.4.2 (Theorem 2 from page 36 of the Textbook)**.** *Let* $f(z) \to A$ *and* $g(z) \to B$ *as* $z \to z_0$ *(where now* $z_0 \in \widehat{\mathbb{C}}$ *to cover also* $z_0 = \infty$*),* $\alpha \in \mathbb{C}$. *Then*

(i) $\alpha f(z) \to \alpha A$, $f(z) + g(z) \to A + B$, $f(z)g(z) \to AB$ *as* $z \to z_0$*;*

(ii) $f(\bar{z})_n \to \bar{A}$, $|f(z)| \to |A|$, $\mathrm{Re}(f(z)) \to \mathrm{Re}(A)$, $\mathrm{Im}(f(z)) \to \mathrm{Im}(A)$ *as* $z \to z_0$*;*

(iii) If $B \neq 0$ *then* $\dfrac{f(z)}{g(z)} \to \dfrac{A}{B}$.

1.4.4 Continuity

Definition 1.4.5. *Function* $f(z)$ *is continuous at point* z_0 if f is defined in z_0 and

$$\lim_{z\to z_0} f(z) = f(z_0).$$

See examples on page 36 of the Textbook.

Theorem 1.4.3 (Theorem 3 from page 37 of the Textbook)**.** *Let $f(z)$ and $g(z)$ be continuous at z_0, $\alpha \in \mathbb{C}$. Then*

(i) $\alpha f(z)$, $f(z) + g(z)$, $f(z)g(z)$ *are continuous at* z_0*;*

(ii) $f(\bar{z})$, $|f(z)|$, $\mathrm{Re}(f(z))$, $\mathrm{Im}(f(z))$ *are continuous at* z_0*;*

(iii) If $g(z_0) \neq 0$ *then* $\dfrac{f(z)}{g(z)}$ *is continuous at* z_0*.*

Corollary 1.4.4. *(i) Polynomial* $P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n$ *is continuous in* $\mathbb{C}$*.*

(ii) Rational function $\dfrac{P(z)}{Q(z)}$ *where* $P(z)$ *and* $Q(z)$ *are two polynomials is continuous in* $\mathbb{C}$ *except of roots of* $Q(z)$*.*

1.4.5 Infinite Series

The notions of the converging, absolutely converging and diverging series repeat those from Calculus I.

Definition 1.4.6. Consider series $\sum_{n=1}^{\infty} a_n$. Then

(a) It is *convergent and its sum its sum is* S if the sequence of *partials sums* $S_N = \sum_{n-1}^{N} a_n$ converges to S.

(b) It is *divergent* if the sequence of partial sums diverges.

See Examples on pages 38–40 of the Textbook.
The following theorem is proven the same way as in Calculus I:

Theorem 1.4.5. *Let series* $\sum_{n=1}^{\infty} a_n$ converges absolutely*, that is*

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

Then it converges.

Power series will play a crucial role in this class.

1.4.6 Some Sums

The geometric series:

$$\sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z}, \qquad z \neq 1$$

$$\implies \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}, \qquad |z| < 1.$$

We can differentiate and integrate it.

1.5 Exponential, Logarithm, and Trigonometric Functions

In this section we introduce *functions of complex variable* e^z, and related hyperbolic and trigonometric functions $\cosh(z)$, $\sinh(z)$, $\cos(z)$ and $\sin(z)$. These functions will coincide with the standard functions of real variable e^x, $\cosh(x)$, $\sinh(x)$, $\cos(x)$ and $\sin(x)$ when $z = x \in \mathbb{R}$, and satisfy the same identities. It will turn out later that these functions are *analytic*.

We also introduce the *inverse functions* $\log(z)$, $\arccos(z)$ and $\arcsin(z)$, which will be *multi-valued functions of complex variable* z, and their *single-valued* variants $\operatorname{Log}(z)$, $\operatorname{Arccos}(z)$ and $\operatorname{Arcsin}(z)$ called *principal branches* of the corresponding multivalued functions.

These functions also coincide with the corresponding functions of the real variable when $z = x \in \mathbb{R}$ and it will turn out later that these functions are analytic.

1.5.1 Exponential Function

We define *exponential function*

$$e^z := e^x e^{iy} = e^x\bigl(\cos(y) + i\sin(y)\bigr)$$

which is completely natural as we want

$$e^{z+w} = e^z e^w \tag{1.5.1}$$

and we already defined $e^{iy} = \cos(y) + i\sin(y)$. Sometimes instead of e^z we write $\exp(z)$. One can check easily, using ordinary identities for cosine and sine of the sum (for real arguments) that (1.5.1) holds.

Further,

$$|e^z| = e^{\operatorname{Re}(z)} \tag{1.5.2}$$

Indeed, $|e^{iy}| = \big(\cos^2(y) + \sin^2(y)\big)^{1/2} = 1 \implies e^{x+iy} = |e^x| \cdot |e^{iy}| = e^x \cdot 1 = e^x$.

Using Wolfram Alpha we can plot $\operatorname{Re}(e^z) = e^x \cos(y)$

https://www.wolframalpha.com/input/?i=Plot(e^xcos(y)).

and similarly $\operatorname{Im}(e^z) = e^x \sin(y)$.

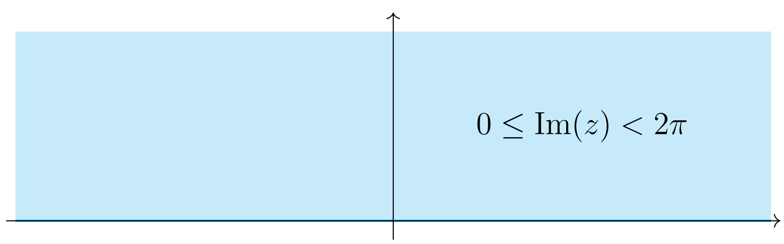
There is also (not very useful) color-coded ComplexPlot

https://www.wolframalpha.com/input/?i=ComplexPlot(e^z)

Let us analyze *complex map* in details. Since this function is $2\pi i$*-periodic*:

$$e^{z+2\pi i} = e^z \tag{1.5.3}$$

we restrict ourselves to the strip $0 \le \operatorname{Im}(z) < 2\pi$:



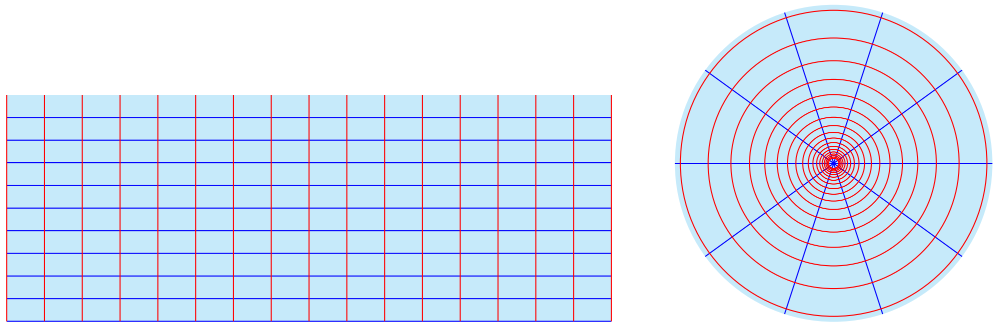
Since $e^z \ne 0$, 0 does not belong to range of e^z but all other values could be achieved. So range of e^z is $\mathbb{C} \setminus \{0\}$.

Observe that the *horizontal lines* $\{z\colon \operatorname{Im}(z) = \mathsf{const}\}$, that is $z = x + yi$ with $-\infty < x < \infty,\ y = \mathsf{const}$, become rays $w = e^x(\cos(y) + i\sin(y))$ that is $\{w\colon \arg(w) = y = \mathsf{const}\}$.

Meanwhile, *vertical segments* $\{z\colon \operatorname{Re}(z) = \mathsf{const},\ 0 < z < 2\pi\}$, that is $z = x + yi$, with $x = \mathsf{const}$, $0 \le y < 2\pi$, become circles $w = e^x(\cos(y) + i\sin(y))$ that is $\{w\colon |w| = e^x\}$.

Remark 1.5.1. (a) When x in the horizontal lines runs from $(-\infty, \infty)$, $|w|$ runs $(0, \infty)$ (so rays are outward).

(b) When y in the vertical segments runs from 0 to 2π, $\arg(w)$ runs $(0, 2\pi)$ (so circles go counter-clockwise).



(a) z (b) $w = e^z$

(c) One can see that *half-strip* $\{z\colon 0 \le \mathrm{Im}(z) < 2\pi,\ \mathrm{Re}(z) < 0\}$ is mapped to the unit disk with a punched center $\{0 < w\colon |w| < 1\}$, while *half-strip* $\{z\colon 0 \le \mathrm{Im}(z) < 2\pi,\ \mathrm{Re}(z) > 0\}$ is mapped to the exterior of the unit disk $\{w\colon |w| > 1\}$.

(d) This function is $2\pi i$-*periodic*: $e^{z+2\pi i} = e^z$. In complex variables periods are not necessary real (may be complex) and there are analytic functions which have two non-proportional complex periods. For example:

https://mathworld.wolfram.com/WeierstrassEllipticFunction.html

(it is way too advanced for our class, we are not covering it, but look at this beauty).

1.5.2 Logarithm Function

We want to consider an inverse function to e^z. From the definition of e^z it follows immediately, that the inverse function $\log(z)$ is defined up to $2\pi in$, $n \in \mathbb{Z}$, thus it is multivalued, and its domain is $\mathbb{C} \setminus \{0\}$ (that is the plane with a punched $\{0\}$),

$$\log(z) \coloneqq \ln(|z|) + i \arg(z). \tag{1.5.4}$$

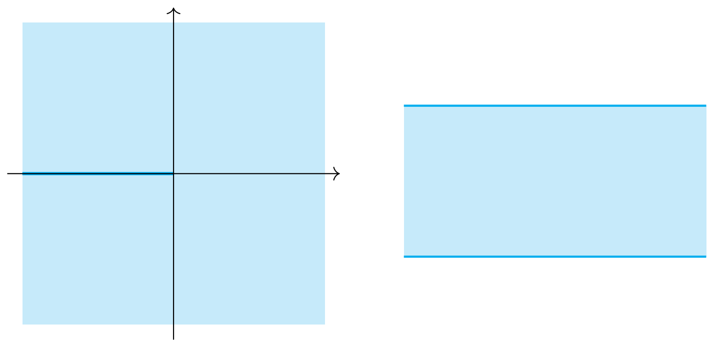
We can make it single-valued

$$\operatorname{Log}(z) := \ln(|z|) + i \operatorname{Arg}(z) \tag{1.5.5}$$

but it will be discontinuous.

Going around the origin once in the counter-clockwise direction adds $2\pi i$ to $\log(z)$, so to make $\operatorname{Log}(z)$ continuous we need not to allow it.

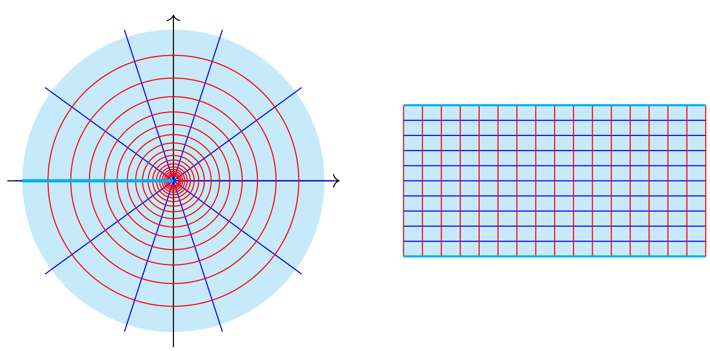
We do it by imposing a *cut* $(-\infty, 0] \subset \mathbb{R}$, that is, we consider $\mathbb{C} \setminus (-\infty, 0]$.



(a) z (b) $w = \operatorname{Log}(z)$

and take $\operatorname{Arg}(z) \in (-\pi, \pi)$ (we are opportunistic here and change the definition as we see fit!); then $\operatorname{Log}(z)$ maps $\mathbb{C} \setminus (-\infty, 0]$ onto strip $\{w\colon -\pi < \operatorname{Im}(w) < \pi\}$.

We selected *a branch* $\operatorname{Arg}(z)$ of $\arg(z)$. *Two different sides of the cut go to two different horizontal straight lines. Cuts always have two sides and in mappings they usually separate.*



(a) z (b) $w = \operatorname{Log}(z)$

Remark 1.5.2. (a) We can take any strip $\{\theta < \operatorname{Im}(z) < \theta + 2\pi\}$ as a domain and we get $\mathbb{C}$ with a cut along $\{z\colon \operatorname{Arg}(z) = \theta\}$ as range.

(b) For map $z \to e^{iz}$ we need to rotate domain by angle $-\frac{\pi}{2}$ (so it will be a vertical strip).

1.5.3 Fractional pPowers

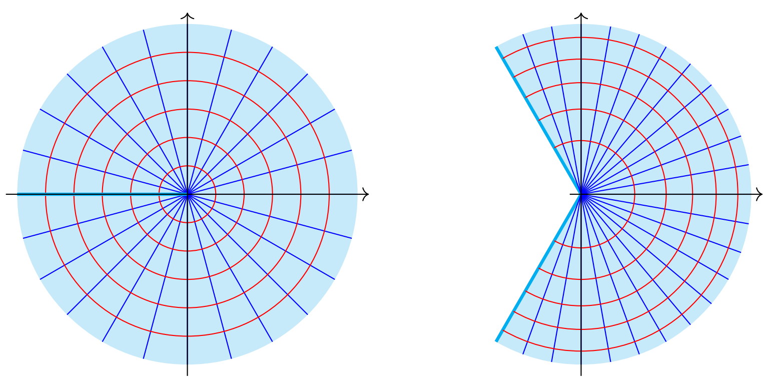
Using logarithm function we can define any power of z:

$$z^\alpha = e^{\alpha \log(z)} \tag{1.5.6}$$

with $\alpha \in \mathbb{C}$. This function is single-valued only when $\alpha \in \mathbb{Z}$ (real integer). Indeed, since $\log(z)$ is defined up to $2\pi i n$, z^α is defined up to a factor $e^{2\pi i \alpha n}$ and it is 1 for all $n \in \mathbb{Z}$ if and only if $\alpha \in \mathbb{Z}$.

Furthermore, if $\alpha \in \mathbb{Q}$, that is $\alpha = p/q$ is a real rational number, with p/q irreducible ratio (so integers p and q have only ± 1 as common divisors, $q > 0$) this is q-valued function, with q different branches, while otherwise it has an infinite number of different branches.

Consider mappings. We want them to be bijective. We consider only real $\alpha \neq 0$. Let us assume that $0 < \alpha < 1$. Let us consider $\mathbb{C} \setminus (-\infty, 0]$ which is mapped to a sector $\{w\colon |\operatorname{Arg}(w)| < \alpha\pi\}$.



(a) z (b) $w = z^\alpha$

Again different sides of the cut are mapped onto different pieces of the boundary. Blue rays and red arcs on both pictures have same orientations.

Remark 1.5.3. (a) Case $\alpha > 1$ is reduced to this one since z^α is inverse to $z^{1/\alpha}$. Then picture (a) and (b) interchange.

(b) One can consider $-1 < \alpha < 0$, pictures remain the same, except the sector now is $\{w\colon |\operatorname{Arg}(w)| < -\alpha\pi\}$ and blue rays and red arcs on both pictures have opposite orientations.

(c) Case $\alpha < -1$ is reduced to the previous one.

Exercise 1.5.1. Think, what happens with in the cases $0 < \alpha < 1$ and $-1 < \alpha <$ with

(a) The disk with a cut $\{z\colon |z| < R,\ |\operatorname{Arg}(z)| < \pi\}$,

(b) The exterior of the disk with a cut $\{z\colon |z| > r,\ |\operatorname{Arg}(z)| < \pi\}$,

(c) The ring with the cut $\{z\colon r < |z| < R,\ |\operatorname{Arg}(z)| < \pi\}$.

1.5.4 Trigonometric Functions

Now we want to introduce *trigonometric functions of complex variable*. They should satisfy the same identities as the corresponding functions of the real variable, and coincide with them as $z = x \in \mathbb{R}$. Let

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2},$$

and

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

because we have these formulas for real z.

Let also

$$\tan(z) = \frac{\sin(z)}{\cos(z)}, \qquad \cot(z) = \frac{\cos(z)}{\sin(z)},$$

and

$$\sec(z) = \frac{1}{\cos(z)} \qquad \csc(z) = \frac{1}{\sin(z)}$$

also because we have these formulas for real z.

Plugging expression for $e^{iz} = e^{-y}(\cos(x) + i\sin(x))$ with $z = x + iy$ we get

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y),$$
$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

with hyperbolic functions of real vatriable

$$\cosh(y) = \frac{e^y + e^{-y}}{2}, \qquad \sinh(y) = \frac{e^y - e^{-y}}{2}.$$

Similarly, $\sec(z) := 1/\cos(z) = (\cos(x)\cosh(y) + i\sin(x)\sinh(y))/(\cosh^2(y) - \sin^2(x))$ and $\operatorname{Re}(\sec(z))$, $\operatorname{Im}(\sec(z))$ can be 3D-plotted by
https://www.wolframalpha.com/input/?i=Plot(cos(x)cosh(y))/(cosh^2(y)-sin^2(x)),
https://www.wolframalpha.com/input/?i=Plot(sin(x)sinh(y))/(cosh^2(y)-sin^2(x))
respectively while plots of $\operatorname{Re}(\csc(z))$, $\operatorname{Im}(\csc(z))$ could be obtained by shifts and/or reflections.

Finally, $\tan(z) = (\sin(2x) + i\sinh(2y))/(\cos(2x) + \cosh(2y))$ and $\operatorname{Re}(\tan(z))$, $\operatorname{Im}(\tan(z))$ are plotted by
https://www.wolframalpha.com/input/?i=Plot(cos(2x)/((cosh^2(y)-sin^2(x)))
https://www.wolframalpha.com/input/?i=Plot(sinh(2y)/((cosh^2(y)-sin^2(x)))
respectively while plots of $\operatorname{Re}(\cot(z))$, $\operatorname{Im}(\cot(z))$ could be obtained by shifts and/or reflections.

Let us consider complex plot of $\cos(z)$ and start from strip $\{-\pi < \operatorname{Re}(z) < \pi\}$ which both e^{iz} and e^{-iz} maps one-to-one on $\mathbb{C} \setminus 0$. However $\cos(-z) = \cos(z)$ and this strip should be cut into two subdomains, so that if z belongs to one of them then $-z$ be;ongs to another.

Let us consider z belonging to the strip $\{z\colon 0 < \operatorname{Re}(z) < \pi\}$: and look where it maps to

(a) Vertical lines $\{z\colon x = \operatorname{Re}(z) = \mathsf{const},\ y = \operatorname{Im}(z)\}$ are mapped to $w = u + iv$ with $v = -\sin(x)\sinh(y)$, $u = \cos(x)\cosh(y)$ which are branches of hyperbolas satisfying

$$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1,$$

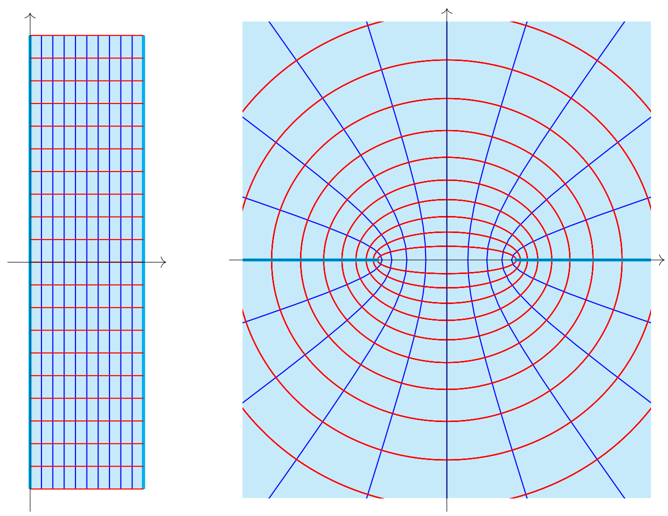
with $A = |\cos(x)|$, $B = \sin(x)$.

(b) Horizontal segments $\{z\colon 0 < x = \operatorname{Re}(z) < \pi,\ y = \operatorname{Im}(z) = \mathsf{const}\}$ are mapped to $w = u + iv$ with $v = -\sin(x)\sinh(y)$, $u = \cos(x)\cosh(y)$

which are halves of ellipses satisfying

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$$

with $a = \cosh(y)$, $b = |\sinh(y)|$.



(a) z (b) $w = \cos(z)$

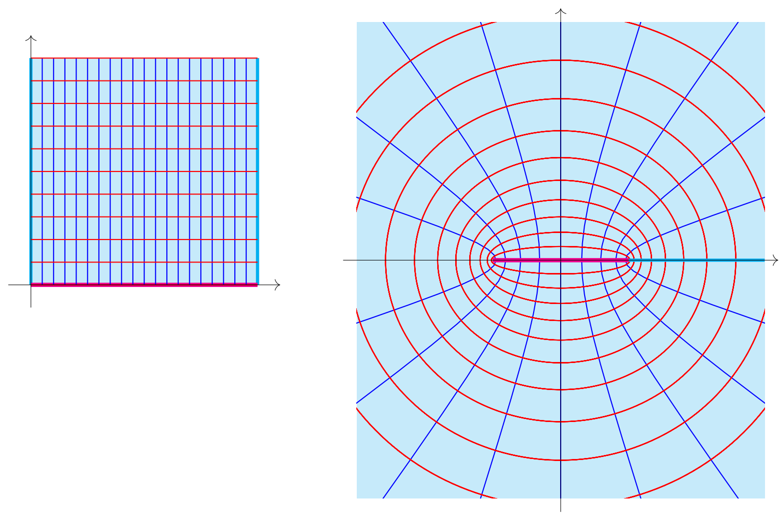
However the same $w = \cos(z)$ restricted to a half-strip $\{z\colon 0 < \operatorname{Re}(z) < 2\pi,\ \operatorname{Im} z > 0\}$, maps it onto the same plane with a different cut: $\mathbb{C} \setminus [-1, \infty]$.

(a) Vertical half-lines $\{z\colon x = \operatorname{Re}(z) = \mathsf{const},\ y = \operatorname{Im}(z)\}$ are mapped to $w = u + iv$ with $v = -\sin(x)\sinh(y)$, $u = \cos(x)\cosh(y)$ which are half-branches of the same hyperbolas as in before.

(b) Horizontal segments $\{z\colon 0 < x = \operatorname{Re}(z) < 2\pi,\ y = \operatorname{Im}(z) = \mathsf{const}\}$ are mapped to $w = u + iv$ with $v = -\sin(x)\sinh(y)$, $u = \cos(x)\cosh(y)$ which are the same ellipses as before.

Similar analysis applies to $\sin(z)$, except strips are shifted and flipped due to $\sin(z) = \cos(\frac{\pi}{2} - z)$.

Finally, $\tan(z) = \frac{\sin(z)}{\cos(z)}$ gives us a picture of Circles of Appolonius with $= i$ and $q = -i$, but explanation why it happens and the discussion of possible domains and ranges will wait until Chapter 3.



(a) z (b) $w = \cos(z)$

1.5.5 Inverse Trigonometric Functions

Consider inverse functions:

$$w = \cos(z) = \frac{e^{iz} + e^{-iz}}{2} \implies e^{2iz} + 1 - 2we^{iz} = 0 \implies$$
$$e^{iz} = w \pm \sqrt{w^2 - 1} \implies iz = \log\bigl(w \pm i\sqrt{1 - w^2}\bigr).$$

Therefore

$$\arccos(w) = -\,i \log\bigl(w \pm \sqrt{1 - w^2}\bigr).$$

Similarly

$$\arcsin(w) = -\,i \log\bigl(iw \pm \sqrt{1 - w^2}\bigr)$$

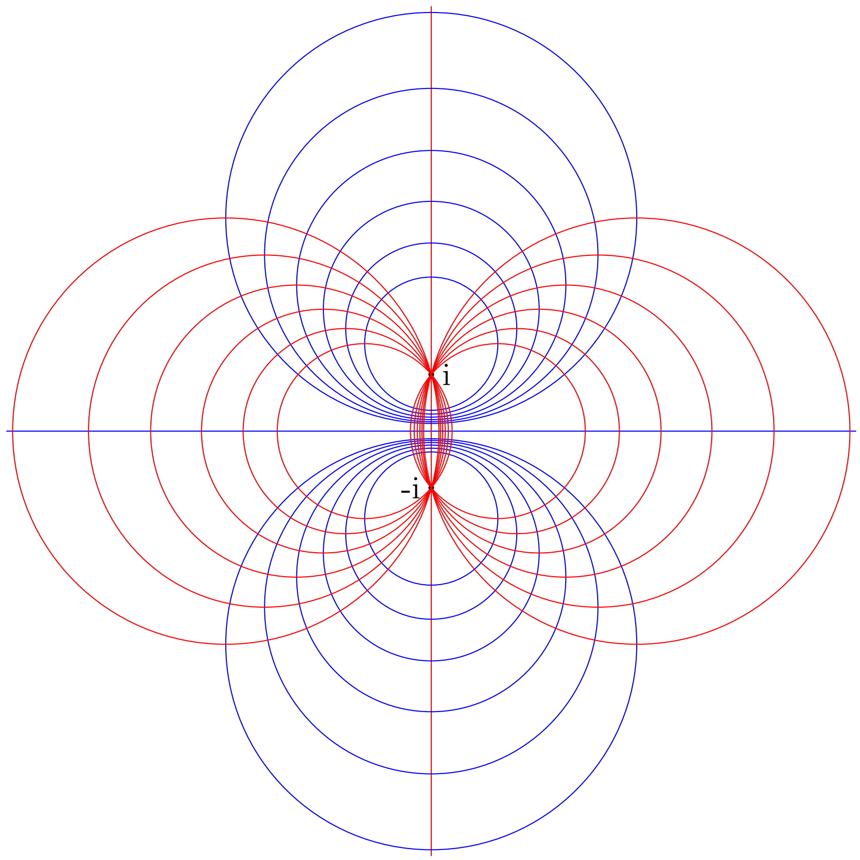
and

$$\arctan(w) = \frac{i}{2} \log\Bigl(\frac{1 - iw}{1 + iw}\Bigr).$$

In particular, range of $\cos(z)$ and $\sin(z)$ in $\mathbb{C}$ (all values are achieved), and range of $\tan(z)$ is $\mathbb{C} \setminus \{i, -i\}$ (all values except $\pm i$ are achieved).

For single-valued functions we have formulas

$$\operatorname{Arccos}(w) = -\,i \log\bigl(w \pm \sqrt{1 - w^2}\bigr),$$
$$\operatorname{Arcsin}(w) = -\,i \log\bigl(iw \ + \sqrt{1 - w^2}\bigr)$$



and

$$\text{Arctan}(w) = \frac{i}{2}\,\text{Log}\Big(\frac{1-iw}{1+iw}\Big)$$

where on the domain (with cuts removed) $\text{Re}(\sqrt{1-w^2}) > 0$, which defines branch uniquely.

Exercise 1.5.2. Check these formulas!

On the other hand, usual formulas hold

$$\begin{aligned}\arccos(w) &= \pm \operatorname{Arccos}(w) + 2\pi n,\\ \arcsin(w) &= (-1)^n \operatorname{Arcsin}(w) + \pi n,\end{aligned}$$

and

$$\arctan(w) = \operatorname{Arctan}(w) + \pi n, \qquad n \in \mathbb{Z}.$$

Exercise 1.5.3. Calculate Arccos(2).

1.5.6 Hyperbolic and Inverse Hyperbolic Functions

We also have *hyperbolic functions*

$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \qquad \sinh(z) = \frac{e^z - e^{-z}}{2}, \qquad \tanh(z) = \frac{\sinh(z)}{\cosh(z)},$$

but due to the formulas

$$\cos(z) = \cosh(iz), \qquad \sin(z) = -i\sinh(iz), \qquad \tanh(z) = -i\tan(iz)$$

their properties and properties and formulas for *inverse hyperbolic functions* follow from the properties of trigonometric functions.

Example 1.5.1. Range of $\tanh(z)$ is $\mathbb{C} \setminus \{1, -1\}$.

1.6 Line Integrals and Green's Theorem

1.6.1 Curves

In this section we simply repeat what was studied in Calculus II, but with the "Complex Variables flavour".

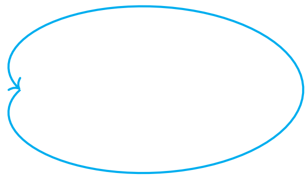
Definition 1.6.1. (a) *A curve* γ is a continuous complex-valued function $\gamma(t)$ defined for t in some interval $[a, b]$ in the real axis; in other words, $\gamma\colon [a, b] \to \mathbb{C}$.

(b) The curve γ is *simple* if

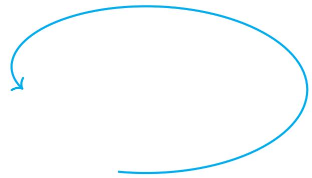
$$a \le t_1 < t_2 \le b \;\&\gamma(t_1) = \gamma(t_2) \implies t_1 = a \;\&\; t_2 = b,$$

that is, it does not have *self-intersections* (but may be closed).

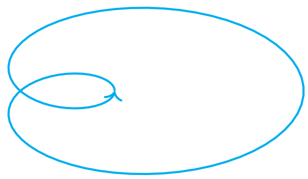
(c) The curve γ is *closed* if $\gamma(a) = \gamma(b)$, that is the *initial point* $\gamma(a)$ coincides with the *end-point* $\gamma(b)$.



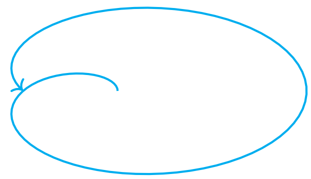
(a) This curve is simple and closed



(b) This curve is simple but not closed



(c) This curve is closed but not simple

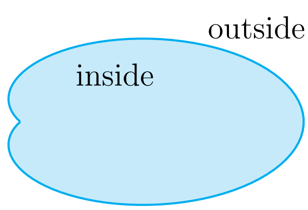


(d) This curve is neither closed nor simple

Remark 1.6.1. The famous *Jordan Curve Theorem* asserts that the complement of the range of a curve, which is simple and closed, consists of two disjoint open connected sets, one *bounded* and the other *unbounded*.

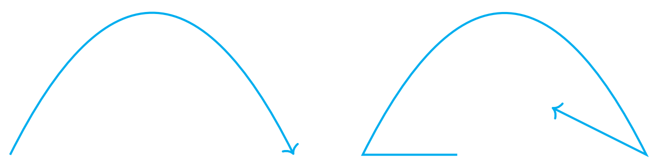
The bounded piece is the *inside of the curve* and the unbounded piece *the outside*.

Despite the almost painful obviousness of this statement, the theorem is hard to prove. We shall accept it as true.



Definition 1.6.2. (a) Curve is *smooth* if $\gamma(t)$ is continuously differentiable (that is, both $\mathrm{Re}(\gamma(t))$ and $\mathrm{Im}(\gamma(t))$ are continuously differentiable).

(b) Curve is *piecewise smooth* if $\gamma(t)$ is piecewise continuously differentiable (that is, there exist $a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$ such that $\gamma(t)$ is continuously differentiable on $[t_0, t_1]$, $[t_1, t_2]$,...,$[t_{n-1}, t_n]$.



(a) This curve is smooth

(b) This curve is piecewise smooth

Remark 1.6.2. (a) It is very common and convenient to refer to the range of $\gamma(t)$ as the curve γ and to $\gamma(t)$ itself as the *parametrization of the curve.*

(b) With this use of the word curve, a curve becomes a concrete geometric object such as a circle or a straight line segment and hence is easily visualized. The difficulty with this view is that a particular curve has many different parameterizations. However, *our results under very broad assumptions would not depend on parametrizations.*

(c) What is more, for closed curves the results would not depend on the choice of start-point (which is also an end-point).

(d) We extend the notion of the curve, requiring $\gamma(t)$ to be only piecewise continuous, that is, consisting of several curves in the old understanding.

(e) Then we can even select a parametrization on each piece separately.

Definition 1.6.3. (a) Standard parametrization of a straight segment from z_0 to z_1: $z(t) = z_0(1-t) + z_1 t$, $0 \le t \le 1$.

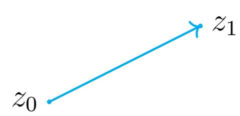
(b) Standard parametrization of a circular arc with a center at c and radius R: $z(t) = c + Re^{it}$ with $\theta_0 < t < \theta_1$.

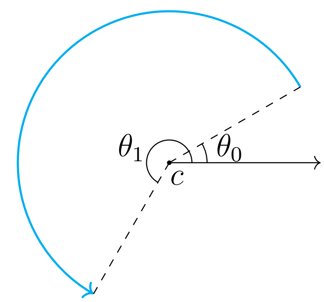
1.6.2 Line Integral

Definition 1.6.4. Let $\gamma = \gamma(t)$ be an oriented smooth curve and let $f(z)$ be a complex-valued function on this curve. Then

$$\int_\gamma f(z)\,dz := \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\,dt$$

is a *line integral of f along γ.*





Remark 1.6.3. This line integral could be written as

$$\int_\gamma f(x(t) + iy(t))\,(dx + idy) := \int_{t_0}^{t_1} f(x(t), y(t))(x'(t) + iy'(t))\,dt$$

where $\gamma(t) = x(t) + iy(t)$, $d\gamma = dx + idy$.

So we reduced the notion of the complex line integral to the notion of the real line integral (complexity of f is not important) and it has the following properties

(a) $\int_\gamma f(z)\,dz$ does not depend on parametrization, that means, if $\gamma'(s) = \gamma(t(s))$ where $t = t(s)$ is continuously differentiable and $t(s_0) = t_0$, $t(s_1) = t_1$ (where now we do not even assume that $t_0 < t_1$ or $s_0 < s_1$ then $\int f(z)\,dz$ does not change.

(b) On the other hand,

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$$

where $-\gamma$ is the same curve but with the *opposite parametrization*:

$$\int_{-\gamma} f(z)\,dz = \int_{t_1}^{t_0} f(\gamma(t))\gamma'(t)\,dt.$$

(c) Also

$$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

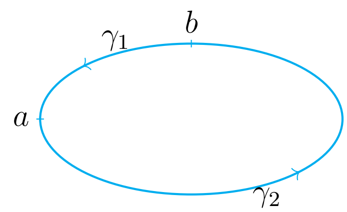
provided $\gamma_1 = \gamma(t)$ with t running from t_0 to t_1, $\gamma_2 = \gamma(t)$ with t running from t_1 to t_2, and $\gamma = \gamma(t)$ with t running from t_0 to t_2.

(d) It allows us to introduce

$$\int_{\gamma} f(z)\,dz := \int_{\gamma_1} f(z)\,dz + \ldots + \int_{\gamma_n} f(z)\,dz$$

for any piecewise smooth curve $\gamma = \gamma_1 + \ldots + \gamma_n$, even consisting of several disjoint segments, because *the right-hand expression does not depend on breaking γ into segments $\gamma_1, \ldots, \gamma_n$.* However, *orientation matters!*

(e) Integral over closed contour does not depend which point is start- and end-point. Indeed, if such points are a and b then the path with start- and end-point a is $\gamma = \gamma_1 + \gamma_2$, and the path with start- and end-point b is $\gamma' = \gamma_2 + \gamma_1$:



(f) For integral over closed contour a special notation often is used: $\oint_{\gamma}$

But recall: orientation matters!

Remark 1.6.4. (a) Line integral differs from *length integral*

$$\int_{\gamma} f(z)\,|dz| := \int_{t_0}^{t_1} f(x(t), y(t))\,\underbrace{\sqrt{(x'(t))^2 + (y'(t))^2}\,dt}$$

where $|dz| = \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$ is a *length element* and *here we assume that $t_0 < t_1$.* We will (almost) never use the length integral in our course.

(b) Recall that length integral also was studied in Calculus II (even in Calculus I) and this integral does not change sign when we change an orientation (which means, parmute star- and end-points).

(c) We can also consider

$$\int_\gamma g(z)\,d\bar{z} := \int_\gamma g(z)\,(dx - idy).$$

We will use the following important inequality, which follows from the standard properties of integral:

Theorem 1.6.1.

$$|\int_\gamma f(z)\,dz| \le \int_\gamma |f(z)|\,|dz|.$$

1.6.3 Green's Formula: Real Variables

Let us recall the following from Calculus II:

Theorem 1.6.2 (Green's formula, real variables)**.**

$$\oint_\gamma \big(M\,dx + N\,dy\big) = \iint_D \big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\big)\,dxdy, \tag{1.6.1}$$

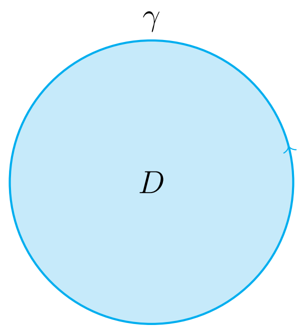
where D is a bounded domain in $\mathbb{R}^2$ and $\gamma = \partial D$ is it's border, properly oriented*, M and N are smooth in the closure of $\bar{D} = D \cup \gamma$ (recall that $\bar{D}$ is a closure of domain D.*

1.6.4 Green's Formula: Discussion

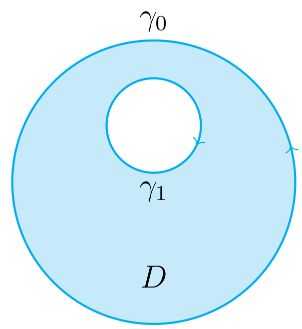
(a) What is properly oriented? If domain D is simply-connected (with no holes inside), then γ is a simple closed curve and it means *counter-clockwise.*

(b) If D is not a simply connected (with holes inside) then $\gamma = \gamma_0 + \gamma_1 + \ldots + \gamma_N$, where γ_0 is the *external boundary, counter-clockwise oriented and* and γ_1 are *inner boundaries* (that means boundaries of the holes), *clock-wise oriented.*

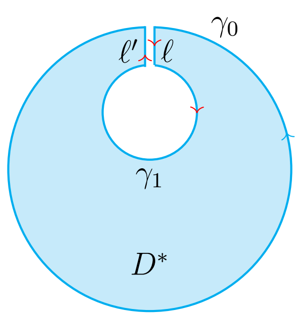
Why? Let us make an *infinitely thin cut* (if $N > 1$ we need N cuts) so that domain D^* after cuts will be connected and simply-connected. Its boundary $\gamma^* = \gamma_0 + \ell + \ell' + \gamma_1$ is connected and the counter-clockwise orientation of γ_0 implies clockwise orientation of γ_1 (and $\gamma_2, \ldots, \gamma_N$) and the proper orientations of ℓ' and ℓ.

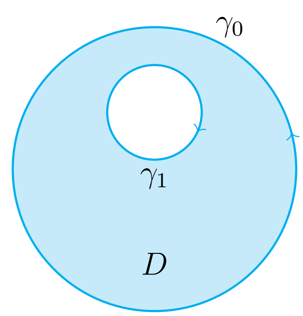


(a) Simply connected domain



(b) Not simply connected domain





Applying to D^*, γ^* Green's formula we see that $\iint_{D^*} = \iint_D$, and $\oint_{\gamma^*} = \int_{\gamma_0} + \int_\ell + \int_{\ell'} + \int_{\gamma_1} = \int_{\gamma_0} + \int_{\gamma_1} = \oint_\gamma$ because $\ell' = -\ell$ (have opposite directions) and therefore $\int_\ell + \int_{\ell'} = 0$, so corresponding integrals cancel one another.

(a) Permute x and y, M and N, then orientation becomes opposite so the left hand expression of (1.6.1) changes sign, and so does the right hand expression.

$$\oint_\gamma \bigl(M\,dx + N\,dy\bigr) = \iint_D \Bigl(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Bigr)\,dxdy. \tag{1.6.1}$$

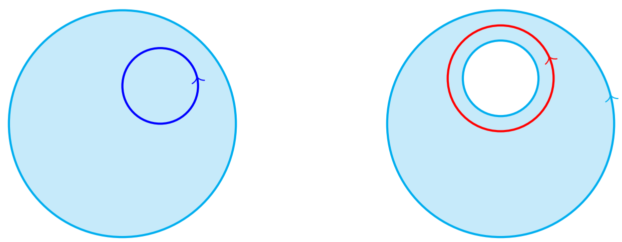
(b) Let

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0. \tag{1.6.2}$$

Then if a closed curve γ' is properly oriented and bounds a domain $D' \subset D$ then

$$\oint_{\gamma'} \bigl(M\,dx + N\,dy\bigr) = \iint_{D'} \Bigl(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Bigr)\,dxdy = 0. \tag{1.6.3}$$

(c) In simple connected domains *every* simple closed curve bounds some subdomain; it is not so in domains which are not simply-connected:

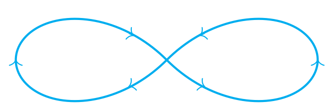


(d) Therefore in simply-connected domains (1.6.2) implies that

$$\oint_{\gamma} \bigl(M\,dx + N\,dy\bigr) = 0 \tag{1.6.4}$$

for any simple closed counter-clockwise oriented curve γ.

(e) We do not need to have γ counter-clockwise oriented or even simple: for clockwise oriented integral will be still 0 since it just changes sign and closed curves with self-intersections could be broken into simple closed curves:

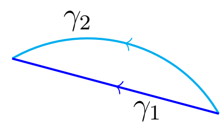


(we slightly cut corners here, since there could be an infinite number of self-intersections).

(f) Then integral over curve which is not closed depends only on it's start and end-points:

Indeed, $\gamma_1 - \gamma_2$ is a closed curve, so

$$0 = \int_{\gamma_1-\gamma_2} = \int_{\gamma_1} - \int_{\gamma_2} \implies \int_{\gamma_1} = \int_{\gamma_2}.$$



(g) Fix point $(x_0, y_0) \in D$ and define

$$U(x,y) = \int_\gamma \bigl(M\,dx + N\,dy\bigr) + C \tag{1.6.5}$$

with an arbitrary constant $C = U(x_0, y_0)$ with integral taken over any curve γ from (x_0, y_0) to (x, y) because in framework of simply-connected D and assumption (1.6.2) this integral does not depend on the choice of γ.

(h) Then $dU = M\,dx + N\,dy$. Indeed

$$dU = \int_{(x_0,y_0)}^{(x+dx,y+dy)} (M\,dx + N\,dy) - \int_{(x_0,y_0)}^{(x,y)} (M\,dx + N\,dy) = \int_{(x,y)}^{(x+dx,y+dy)} (M\,dx + N\,dy) = M\,dx + N\,dy.$$

Therefore

Theorem 1.6.3. *If domain D is simply-connected and $M\,dx + N\,dy$ is* closed *that means*

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0. \tag{1.6.2}$$

then $M\,dx+N\,dy$ is exact *that means for some $M\,dx+N\,dy = dU$ for some function U and therefore*

$$M = \frac{\partial U}{\partial x}, \qquad N = \frac{\partial U}{\partial y}. \tag{1.6.6}$$

Remark 1.6.5. (a) U is defined up to constant C.

(b) Conversely, if $M\,dx + N\,dy$ is exact, then it is closed (and we do not need D to be simply-connected.

(c) On the other hand, if D is not simply-connected, then some closed forms $M\,d + N\,dy$ are not exact.

Example 1.6.1. Indeed, consider $\mathbb{C}\setminus\{0\}$ and

$$\frac{x\,dy - y\,dx}{x^2+y^2} = d\,\theta, \qquad \theta = \arg(x+yi);$$

so it is closed. However polar angle $\arg(x+yi)$ is not a single-valued function on $\mathbb{C}\setminus\{0\}$ and

$$\oint_\gamma \frac{x\,dy - y\,dx}{x^2+y^2} = 2\pi$$

where γ winds exactly once in the counter-clockwise direction around 0.

1.6.5 Green's Formula: Complex Variables

Theorem 1.6.4 (Green's theorem, complex variables)**.**

$$\oint_\gamma \bigl(f\,dz + g\,d\bar{z}\bigr) = 2i\iint_D \Bigl(\frac{\partial f}{\partial \bar{z}} - \frac{\partial g}{\partial z}\Bigr)\,dxdy, \tag{1.6.7}$$

where D is a bounded domain in $\mathbb{C}$ and $\gamma = \partial D$ is it's border, properly oriented, f and g are smooth in $\bar{D} = D\cup\gamma$ and

$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\Bigl(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Bigr), \tag{1.6.8}$$

$$\frac{\partial g}{\partial z} := \frac{1}{2}\Bigl(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\Bigr). \tag{1.6.9}$$

Proof. Indeed, we need to apply "real" Green's formula; since

$$\begin{aligned} f\,dz + g\,d\bar{z} &= f(dx+idy) + g(dx - idy)\\ &= (f+g)\,dx + i(f-g)\,dy \end{aligned}$$

we need to plug $M = f+g$, $N = i(f-g)$, so we need to calculate

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial (if - ig)}{\partial x} - \frac{\partial (f+g)}{\partial y} = i\Bigl(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Bigr) - i\Bigl(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\Bigr).$$

□

Remark 1.6.6. Why these notations (1.6.8) and (1.6.9)? Let us write

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{1.6.10}$$

On the other hand $dz = dx + idy$, $d\bar{z} = dx - idy$ imply that $dx = \frac{1}{2}(dz + d\bar{z})$ and $dy = -\frac{i}{2}(dz - d\bar{z})$ and therefore

$$df = \frac{1}{2}\big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\big)dz + \frac{1}{2}\big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\big)\,d\bar{z}$$

and in these notations we extend the usual formula to complex variables:

$$df = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar{z}}\,d\bar{z} \tag{1.6.11}$$

even if $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ *are not partial derivatives in the sense of Calculus II.*

We will use only

Corollary 1.6.5.

$$\oint_\gamma f\,dz = 2i\iint_D \frac{\partial f}{\partial \bar{z}}\,dxdy, \tag{1.6.12}$$

where D is a bounded domain in $\mathbb{C}$ and $\gamma = \partial D$ is it's border, properly oriented.

Remark 1.6.7. If you took MAT257 (Analysis II= Calculus II on steroids), then you know that

(a) $M\,dx + N\,dy$ is called 1-*form*;

(b) one can write $dxdy$ as $dx \wedge dy$.

(c) On the other hand $dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy$ and therefore one can rewite (1.6.7) as

$$\oint_L \big(f\,dz + g\,d\bar{z}\big) = -\iint_D \big(\frac{\partial f}{\partial \bar{z}} - \frac{\partial g}{\partial z}\big)\,dz \wedge d\bar{z}. \tag{1.6.13}$$

This is Green's formula in (perfectly) complex form.

Chapter 2

Basic Properties of Analytic Functions

2.1 Analytic and Harmonic Functions; Cauchy-Riemann Equations

2.1.1 Analytic functions

Now we want to introduce our main definition.

Recall formulas (1.6.11), (1.6.8), (1.6.9):

$$df = \frac{\partial f}{\partial z}\, dz + \frac{\partial f}{\partial \bar{z}}\, d\bar{z}. \tag{2.1.1}$$

with

$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big), \tag{2.1.2}$$

$$\frac{\partial f}{\partial z} := \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big). \tag{2.1.3}$$

To prove (2.1.1)–(2.1.3) plug $dx = \frac{1}{2}(dz + d\bar{z})$, $dy = \frac{1}{2i}(dz - d\bar{z})$ into usual formula from Calculus II

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy \tag{2.1.4}$$

and arrive to

$$df = \frac{1}{2}\frac{\partial f}{\partial x}(dz + d\bar{z}) + \frac{i}{2}\frac{\partial f}{\partial y}(d\bar{z} - dz)$$
$$= \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial x})dz + \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial x})d\bar{z}.$$

Do these two formulas

$$df = \frac{\partial f}{\partial z}\, dz + \frac{\partial f}{\partial \bar{z}}\, d\bar{z} \tag{2.1.1}$$

and

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy \tag{2.1.4}$$

Look the same? Think again!

Think again! In (2.1.4) x and y are *independent variables*, so dx and dy are independent, but in (2.1.1) z and $\bar{z}$ are *not independent*, and dz and $d\bar{z}$ are *not independent*. In fact, z uniquely defines $\bar{z}$, and dz uniquely defines $d\bar{z}$.

However,

Lemma 2.1.1. *dz and $d\bar{z}$ are* linearly independent*: for any complex numbers A and B*

$$A\, dz + B\, d\bar{z} = 0 \quad \forall dz \implies A = B = 0.$$

Proof. Indeed, plugging $dz = dx + idy$, $d\bar{z} = dx - idy$ we get

$$(A + B)\, dx + i(A - B)\, dy = 0 \qquad \forall dx, dy$$

which implies $A + B = A - B = 0$ and then $A = B = 0$. □

Therefore,

Corollary 2.1.2.

$$df = A\, dz + Bd\bar{z} \qquad \forall dz$$

implies that $A = \frac{\partial f}{\partial z}$, $B = \frac{\partial f}{\partial \bar{z}}$ *defined by* (2.1.2)–(2.1.3).

However, we want to have a formula from Calculus I:

$$df(z) = f'(z)\,dz \tag{2.1.5}$$

and we arrive to our main definition:

Definition 2.1.1. (a) Function $f(z)$ is *analytic in the domain* $D \subset \mathbb{C}$ if $df(z) = A(z)\,dz$ holds for all dz and all $z \in D$.

(b) Then we call $A(z)$ *derivative* of analytic $f(z)$ function $f(z)$ and denote it by $f'(z)$ and $\frac{df}{dz}$.

(c) Function analytic in $\mathbb{C}$ is called *entire analytic function* or, simply, *entire function*.

Then we have:

Theorem 2.1.3. *$f(z)$ is analytic in domain D if and only if there*

$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\Bigl(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Bigr) = 0, \tag{2.1.6}$$

in which case (2.1.5) *holds with*

$$f'(z) = \frac{df}{dz} := \frac{\partial f}{\partial z}. \tag{2.1.7}$$

Remark 2.1.1. It is equivalent to definition on page 77 of the Textbook:

(a) A function f defined for z in a domain D is *differentiable at a point z_0 in D* if the following limit exists:

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

(b) This limit, if it exists, is denoted by $f'(z_0)$.

(c) If f is differentiable at each point of the domain D, then f is called *analytic in D*.

(d) A function analytic on the whole complex plane is called *entire*.

Remark 2.1.2. Analytic functions are often called *holomorphic*.

Theorem 2.1.4. *Let f and g be analytic in domain D. Then*

(i) αf *is analytic in D ($\alpha \in \mathbb{C}$ and* $(\alpha f)' = \alpha f'$;

(ii) $f + g$ *is analytic in D and* $(f+g)' = f' + g'$;

(iii) fg *is analytic in D and* $(fg)' = f'g + fg'$;

(iv) $\frac{f}{g}$ *is analytic in* $\{z \in D\colon g(z) \neq 0\}$ *and* $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

Proof. The proof is the same as in Calculus I: f.e.

$$d(fg) = g\,df + f\,dg = gf'\,dz + fg'\,dz = (gf' + fg')\,dz,$$

which implies Statement (iii). □

Theorem 2.1.5. *Let f be analytic in D and g be analytic in G, where* $z \in D \implies f(z) \in G$.
Then $g \circ f$ is analytic in D and $(g \circ f)'(z) = (g' \circ f)f'$.

Proof. Indeed, like in Calculus I, let $w = f(z)$, then

$$dg(w) = g'(w)\,dw = g'(w) \cdot f'(z)\,dz.$$

□

Example 2.1.1. Polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

is entire analytic function and

$$P'(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \ldots + a_{n-1}.$$

Proof. Proof follows from Theorem 2.1.4. □

Example 2.1.2. A rational function $\frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials, is an analytic function on the set $\{z\colon Q(z) \neq 0\}$.

Example 2.1.3. e^z is an entire analytic function and $(e^z)' = e^z$.

Proof. Indeed

$$\begin{aligned}de^z =& d\big[e^x\big(\cos(y)+i\sin(y)\big)\big]\\ =&\big(\cos(y)+i\sin(y)\big)e^x\,dx + e^x\big(-\sin(y)+i\cos(y)\big)\,dy\\ =&e^x\big(\cos(y)+i\sin(y)\big)\,dx + e^x\big(\cos(y)+i\sin(y)\big)i\,dy\\ =&e^z\,(dx+idy) = e^z\,dz.\end{aligned}$$

□

Example 2.1.4. (a) $\cos(z)$, $\sin(z)$, $\cosh(z)$ and $\sinh(z)$ are entire analytic functions and usual expression for derivatives hold.

(b) $\tan(z)$, $\sec(z)$ are analytic functions except when $\cos(z)=0$ (so except $\{z=\frac{\pi}{2}+\pi n,\ n\in\mathbb{Z}\}$) and usual expression for derivatives hold.

(c) $\cot(z)$, $\csc(z)$ are analytic functions except when $\sin(z)=0$ (so except $\{z=\pi n,\ n\in\mathbb{Z}\}$ and usual expression for derivatives hold.

(d) $\tanh(z)$, is analytic function except when $\cosh(z)=0$ (so except $\{z=\frac{i\pi}{2}+i\pi n,\ n\in\mathbb{Z}\}$) and usual expression for derivatives hold.

Example 2.1.5. $\operatorname{Log}(z)$ is an analytic function on $\mathbb{C}\setminus[0,\infty)$ (domain with a cut) and $(\operatorname{Log}(z))'=\frac{1}{z}$.

Proof. Indeed, $w=\operatorname{Log}(z)$ is defined there and $z=e^w$. Then $dz=e^w\,dw=z\,dw \implies dw=\frac{dz}{z}$. □

Example 2.1.6. z^α is an analytic function on $\mathbb{C}\setminus[0,\infty)$ (domain with a cut) and $(z^\alpha)'=\alpha z^{\alpha-1}$.

Proof. Indeed, $z^\alpha=e^{\alpha\operatorname{Log}(z)}$ is defined there and

$$\big(z^\alpha\big)'=\big(e^{\alpha\operatorname{Log}(z)}\big)'=e^{\alpha\operatorname{Log}(z)}\times\alpha\times z^{-1}=\alpha z^\alpha\times z^{-1}=\alpha z^{\alpha-1}.$$

□

Remark 2.1.3. In these examples we can make a cut along any ray from 0 to ∞ and select a corresponding branch.

Later we will prove

Theorem 2.1.6 (Inverse function theorem)**.** *Let f be an analytic function in the vicinity of z_0 and $f'(z_0)\neq 0$. Let $w_0=f(z_0)$.*

Then in the vicinity of w_0 is uniquely determined inverse function $g(w)$, $f(g(w))=w$ and $g(f(z))=z$, and $g'(w)=\dfrac{1}{f'(g(w))}$.

Then we will prove analyticity of $\operatorname{Arcsin}(z)$ and $\operatorname{Arccos}(z)$ in $\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$ and $\operatorname{Arctan}(z)$ and $\operatorname{Arccot}(z)$ in $\mathbb{C}\setminus((-i\infty,-i]\cup[i,i\infty))$ and usual formulas for derivatives.

(a) Domain of Arcsin and Arccos

(b) Domain of Arctan and Arccot

2.1.2 Cauchy-Riemann equations

In the previous Subsection we derived *Cauchy-Riemann equation in the complex form* (also we have not called them so):

$$\frac{\partial f}{\partial \bar z}:=\frac{1}{2}\Big(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\Big)=0. \tag{2.1.8}$$

Since $f(z)=u(x,y)+iv(x,y)$ with $u=\operatorname{Re}(f)$ and $v=\operatorname{Im}(f)$ (here we write argument z as a pair (x,y)) we get

$$\Big(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\Big)(u+iv)=\Big(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\Big)+i\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big)$$

and therefore (2.1.8) (which is equivalent to analyticity of f) is equivalent to the pair of real equations. Thus we arrive to the following theorem:

Theorem 2.1.7. *Function $f=u+iv$ is an analytic function of the complex variable z if and only if u and v satisfy the following* Cauchy-Riemann equations in the real form *(or simply Cauchy-Riemann equations):*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{2.1.9}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.1.10}$$

Differentiating the first equation by x and the second by y (or by y and x and substracting) and adding we get

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{2.1.11}$$

$$\Delta v := \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{2.1.12}$$

Definition 2.1.2. Operator $\Delta = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ is called *Laplace operator* (or *Laplacian*), equation $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is called *Laplace equation*, and functions satisfying it *harmonic functions*.

Therefore we have proven

Theorem 2.1.8. *The real and imaginary parts of complex analytic function f are harmonic functions. Also f itself is a harmonic function.*

Definition 2.1.3. Harmonic functions u and v, satisfying Cauchy-Riemann equations, are called *conjugate harmonic functions*.

Question. Knowing harmonic function u (or v) can we restore a conjugate harmonic function v (u respectively) and therefore an analytic function f?

To answer this question assume that we know u and look at equations (2.1.9) and (2.1.10):

$$\frac{\partial v}{\partial x} = M := -\frac{\partial u}{\partial y}, \qquad \frac{\partial v}{\partial y} = N := \frac{\partial u}{\partial x}.$$

We know that it is necessary to have $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$ and in the simply-connected domain it is also sufficient. Since $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \Delta u$ we arrive to the following theorem:

Theorem 2.1.9. *Let u be a harmonic function in D. Then* *(i) If D is simply-connected then there exists a conjugate harmonic function v.*

(ii) This function v is defined up to additive constant.

It immediately implies

Corollary 2.1.10. *(i) In the simply-connected domain an analytic function could be restored by its real (or imaginary) part.*

(ii) This analytic function is defined uniquely up to an imaginary (correspondingly real) additive constant.

Example 2.1.7. One can check easily that $u = x^3 - 3xy^2$ is a harmonic function in $\mathbb{C}$. To find a conjugate harmonic function we write

$$\frac{\partial v}{\partial x} = 6xy, \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2.$$

Let us recall how we solve such overdetermined systems in MAT244 ODE: Integrating the first equation by x we get $v = 3x^2y + h(y)$ with unknown function $h(y)$. Plugging it to the second equation we get $h'(y) = -3y^2$. Therefore $h(y) = -y^3 + C$. Then $v(y) = 3x^2y - y^3 + C$ and

$$f = x^3 - 3xy^2 + i(3x^2y - y^3 + C) = (x + iy)^3 + Ci = z^3 + Ci.$$

Example 2.1.8. Consider multivalued analytic function $f(z) = \log(z) := \ln|z| + i\arg(z)$ in $\mathbb{C} \setminus \{0\}$. It's real part is $\ln|z| = \frac{1}{2}\ln(x^2 + y^2)$. Since f is analytic its real part $u = \frac{1}{2}\ln(x^2 + y^2)$ must be harmonic in $\mathbb{R}^2 \setminus \{0\}$, but you can check it independently.

If we consider $\mathbb{C}$ with a proper cut (from 0 to ∞ along some ray), we can select here a branch $v = \arg(x + iy)$ as a single valued harmonic function. F.e. if we make a cut $\{(0, y)\colon y \le 0\}$ we can select $v = \operatorname{Arctan}(y/x)$ for $x > 0$ and $v = \pi + \operatorname{Arctan}(y/x)$ for $x < 0$ and $v = \pi/2$ for $x = 0, y > 0$ (but there will be a jump at $x = 0, y < 0$).

However, a single valued function v (and thus f) does not exist. One can explain it calculating

$$\oint_\gamma (M\,dx + N\,dy) = \oint_\gamma \frac{x\,dy - y\,dx}{x^2 + y^2} = \int_0^{2\pi} d\theta = 2\pi \ne 0.$$

where γ is a circular curve centered in 0 and counter-clockwise oriented.

Finally, let us prove

Theorem 2.1.11. *Let f be an analytic function in D. Then*

(i) $\mathrm{Re}(f) = \mathsf{const} \implies f = \mathsf{const}$ *and* $\mathrm{Im}(f) = \mathsf{const} \implies f = \mathsf{const}$.

(ii) $|f| = \mathsf{const} \implies f = \mathsf{const}$ *and* $\arg(f) = \mathsf{const} \implies f = \mathsf{const}$.

Proof. (i) Indeed, if $u = \mathrm{Re}(f) = \mathsf{const}$, then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ and from Cauchy-Riemann equations we see $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$, and $v = \mathsf{const}$.

The second part of Statement (i) is proven in the same way.

(ii) If $|f| = 0$ then $f = 0$. Let $|f| = c \neq 0$. Then $g := \log(f)$ is an analytic function with a constant real part $\ln(|f|)$ and therefore according to Statement **1** it is constant: $g = \mathsf{const}$. Then $f = e^g = \mathsf{const}$.

The second part of Statement (ii) is proven in the same way.

□

2.2 Power series

2.2.1 Definition

Definition 2.2.1. *A power series* is an infinite series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{2.2.1}$$

where $a_0, a_1, \ldots$ are complex numbers, called *coefficients* of the series, z_0 is a fixed point called *a center of the series.*

Example 2.2.1. (a) A polynomial

$$P(z) = \sum_{n=0}^{N} a_n z^n$$

(b) Geometric series

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \qquad |z| < 1.$$

(c) Other important series, converging everywhere

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

Theorem 2.2.1. *Suppose there is some $z_1 \neq z_0$ such that the series* (2.2.1) *converges.*

Then this series converges absolutely and uniformly in every disk $\{z\colon |z - z_0| \le r\}$ with $r < |z_1 - z_0|$.

Proof. Since series $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges, then

$$\lim_{n\to\infty} |a_n|\,|z_1 - z_0|^n = 0.$$

In particular, for some constant M

$$|a_n|\,|z_1 - z_0|^n \le M \qquad \forall n$$

and therefore

$$|a_n|\,|z - z_0|^n \le M\rho^n, \qquad \rho = \frac{|z - z_0|}{|z_1 - z_0|}$$

for all n, with $\rho < 1 \iff |z - z_0| \le r < |z_1 - z_0|$.

Since $\sum_{n=0}^{\infty} M\rho^n < \infty$, our series converges absolutely and uniformly. □

2.2.2 Radius and disk of convergence

Therefore, there are exactly three mutually exclusive cases:

Case 1. Series

$$S(z) := \sum_{n=1} a_n (z - z_0)^n \tag{2.2.1}$$

converges for $z = z_0$ only.

Case 2. Series (2.2.1) converges for all $z \in \mathbb{C}$.

Case 3. Series (2.2.1) converges for some $z \neq z_0$ but not for all of them.

Explore Case 3. So, there are $z_1 \neq z_0$ for each series (2.2.1) converges, and $z_2 \neq z_0$ for which it diverges.

By Theorem 2.2.1 it is impossible that $|z_1 - z_0| > |z_2 - z_0|$ and therefore $|z_1 - z_0| \le |z_2 - z_0|$. Therefore, there exists $R\colon 0 < R < \infty$ such that

- For all $z\colon |z - z_0| > R$ series (2.2.1) diverges.
- For all $z\colon |z - z_0| < R$ series (2.2.1) converges.

Here R is the largest number such that $|z - z_0| < R \implies$ convergence. Also R is the smallest number such that $|z - z_0| > R \implies$ divergence.

Definition 2.2.2. (a) In Case 1 (series converges for $z = z_0$ only) the *radius of convergence* is $R = 0$.

(b) In Case 2 (series converges for all $z \in \mathbb{C}$) the *radius of convergence* is $R = \infty$ and the *disk of convergence* is $\mathbb{C}$.

(c) In Case 3 the *radius of convergence* is R $(0 < R < \infty)$ and the *disk of convergence* is $\{z\colon |z - z_0| < R\}$.

Further, in the disk $\{z\colon |z - z_0| \le r\}$ converges uniformly and absolutely for any $r < R$.

Theorem 2.2.2. *Consider a power series* (2.2.1) *with the radius of convergence R.*

(i) If the limit on the right side exists, then

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}. \tag{2.2.2}$$

(ii) If the limit on the right side exists, then

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|}. \tag{2.2.3}$$

Proof. Proof follows from the *ratio test* and *root test* for numerical series. □

Example 2.2.2. (a) The radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$ is $R = \infty$ and it converges in $\mathbb{C}$.

(b) The radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n(z+1)^n}{5^n}$ is $R = 5$ and the disk of convergence is $\{z\colon |z+1| < 5\}$.

(c) The radius of convergence of $\displaystyle\sum_{n=0}^{\infty} n!!(z-i)^n$ is $R = 0$ and it converges only in $\{i\}$.

Question. What happens on the border of the disk of convergence $\{z\colon |z-z_0| = R\}$ (when $0 < R < \infty$)?

Remember, in Calculus I, where power series of $x \in \mathbb{R}$ were considered, there was an *interval of convergence* $(x_0 - R, x_0 + R)$ and the boundary consisted of two extreme points, $x_0 - R$, and $x_0 + R$. Then you investigated these two extreme points.

Now question is way more difficult, because *the boundary of the disk* $\{z\colon |z - z_0| < R\}$ *is a circle* $\{z\colon |z - z_0| = R\}$, not just two points.

Example 2.2.3. Consider the following series. One can check easily, that all of them have the radius of convergence, equal 1. So, consider $C = \{z\colon |z| = 1\}$.

(a) $\sum_{n=1}^{\infty} z^n$ diverges for all $z\colon |z| = 1$ (common term does not tend to 0).

(b) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges for all $z\colon |z| = 1$ ($\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$).

The next examples are much more difficult:

(c) $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges for all $z \in C, z \neq 1$ and diverges for $z = 1$. One can prove easily the convergence as $z = -1$.

(d) Then $\sum_{n=1}^{\infty} \frac{z^{pn}}{n}$ with $p = 2, 3, \ldots$ diverges for $z_m = e^{2\pi mi/p}$ with $m = 0, 1, \ldots, p-1$ (which are roots of degree p from 1) and converges for all other $z \in C$.

So, the correct and complete answer to the Question (in the Test 1) Find where series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges should indicate that

(a) it converges as $|z| < 1$,

(b) diverges as $|z| > 1$,

(c) converges as $z = -1$, diverges as $z = 1$, and *there are other points in* $z\colon |z| = 1$ *but you don't know about convergence in these points.*

The last part of this answer is important, because it shows, that you understand, *that the boundary of the disk is a circle, not just two points.*

2.2.3 Analyticity of the Sum a Power Series

Theorem 2.2.3. *Assume that a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{2.2.4}$$

converges in the disk $D = \{z\colon |z - z_0| < R\}$, $R > 0$. *Then*

(i) $f(z)$ *is an analytic function in this disk* D;

(ii) $f(z)$ *is differentiable in this disk* D *and*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}. \tag{2.2.5}$$

this series also converges in D.

Proof. Without any loss of the generality we can assume that $z_0 = 0$. Consider $z\colon |z| < r = R - 2\delta$ and $h\colon |h| < \delta$. Then for $|z| \le r = R - 2\delta$

$$\begin{aligned}
|\Big(\frac{(z+h)^n - z^n}{h} - nz^{n-1}\Big)| =& |\sum_{j=2}^{n} \binom{n}{j} z^{n-j} h^{j-1}| \\
\le& |h|\delta^{-2} \sum_{j=2}^{n} \binom{n}{j} (R - 2\delta)^{n-j} \delta^j \\
<& |h|\delta^{-2} \sum_{j=0}^{n} \binom{n}{j} (R - 2\delta)^{n-j} \delta^j \\
=& |h|\delta^{-2}\big((R - 2\delta) + \delta\big)^n = |h|\delta^{-2}(R - \delta)^n.
\end{aligned}$$

Then

$$|\frac{f(z+h) - f(z)}{h} - g(z)| \le |h|\delta^{-2} \sum_{n=2}^{\infty} |a|_n (R - \delta)^n$$

where $g(z)$ is the right-hand expression of (2.2.5). Since $|a_n| \le M(R-\delta/2)^{-n}$ for some $M = M(\delta)$, the right-hand expression does not exceed $M_1|h| \to 0$ as $|h| \to 0$.

We claim that series $g(z)$ converges. Indeed,

$$\sum_{n=1}^{\infty} n|a_n||z-z_0|^{n-1} \le M\delta^{-1}\sum_{n=1}^{\infty} n\rho^n < \infty$$

with $\rho = (R-2\delta)/(R-\delta)$.

Then $f(z)$ is an analytic function and $f'(z) = g(z)$. □

Corollary 2.2.4. *Assume that a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \tag{2.2.4}$$

converges in the disk $D = \{z\colon |z-z_0| < R\}$, $R > 0$. *Then*

(i) $f(z)$ *is infinitely differentiable in* D *and*

$$f^{(m)}(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} a_n (z-z_0)^{n-m}. \tag{2.2.6}$$

(ii) In particular,

$$f^{(m)}(z_0) = m! a_m. \tag{2.2.7}$$

2.2.4 Primitive of the sum of a power series

Corollary 2.2.5. *Assume that a power series* (2.2.4) *converges in the disk* $D = \{z\colon |z-z_0| < R\}$, $R > 0$. *Then*

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-z_0)^{n+1} \tag{2.2.8}$$

converges in D *and is a primitive function of* $f(z)$: $F'(z) = f(z)$ *and* $F(z_0) = 0$.

Example 2.2.4. (a) Integrating $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, converging as $|z| < 1$, we get

$$\operatorname{Log}(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

(b) Integrating $\dfrac{1}{1+z^2} = \sum_{n=0}^{\infty}(-1)^n z^{2n}$, converging as $|z|<1$, we get

$$\operatorname{Arctan}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1}$$

(c) Consider $(1-z)^{-\alpha}$; it's Taylor decomposition is

$$(1-z)^{-\alpha} = 1 + \alpha z + \frac{\alpha(\alpha+1)}{2} z^2 + \dots \\ + \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} z^n + \dots \tag{2.2.9}$$

Unless α is a non-positive integer we get an infinite series. In particular, plugging $\alpha = \frac{1}{2}$ and $z := z^2$ we get a decomposition for $\dfrac{1}{\sqrt{1-z^2}}$. Integrating, we get a decomposition for $\operatorname{Arcsin}(z)$.

Remark 2.2.1. One can prove that

(a) In Series (2.2.9) coefficient a_n has a magnitude $n^{\alpha-1}$ and therefore it converges absolutely on $C = \{z\colon |z|=1\}$ for $\alpha<0$; diverges on C for $\alpha \ge 1$ and converges on C except $z=1$ for $0<\alpha<1$;

(b) Series for $\operatorname{Log}(1-z)$ converges on C except $z=1$;

(c) Series for $\operatorname{Arctan}(z)$ converges on C except $z=\pm i$;

(d) Series for $\operatorname{Arcsin}(z)$ converges on C but not absolutely.

Results "converges but not absolutely" are more difficult and beyond our reach.

2.2.5 Multiplication and division of powers series

Theorem 2.2.6. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \textit{and} \quad g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

converge in the disk $D = \{z\colon |z - z_0| < R\}$*. Then*

$$(fg)(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{2.2.10}$$

with

$$c_n = \sum_{m=0}^{n} a_m b_{n-m} \tag{2.2.11}$$

also converges in D*.*

On the other hand, let us

$$f(z) = \sum_{n=0}^{\infty} c_n (x - z_0)^n, \qquad g = \sum_{n=0}^{\infty} a_n (x - z_0)^n \tag{2.2.12}$$

in $D = \{z\colon |z - z_0| < R\}$. Assume that

$$|g(z)| > 0 \qquad \text{in} \quad D. \tag{2.2.13}$$

Then in D

$$h(z) := f(z)/g(z) = \sum_{n=0}^{\infty} b_n (x - z_0)^n \tag{2.2.14}$$

where b_n could be found recurrently from (2.2.11). In particular, $b_0 = c_0 a_0^{-1}$.

2.3 Cauchy's Theorem and Cauchy's Formula

2.3.1 Cauchy's theorem

Together with Definition of analyticity Cauchy's Theorem is the linchpin of complex variables.

We shall first prove Cauchy's Theorem with the added assumption that f' is continuous. As it turns out, an analytic function automatically has a continuous derivative. However, it is somewhat technical to prove this.

For that reason, most of this work is isolated as a special starred Subsection 2.3.1* of the Textbook; it is optional and we do not cover this in our Notes.

The final step is Theorem 1 of Section 4 (also not covered). Throughout this section, *we shall assume that* f' *is continuous.*

Theorem 2.3.1 (Cauchy's Theorem). *Suppose that f is analytic on a domain D. Let γ be a piecewise simple smooth counter-clockwise oriented closed curve in D whose inside Q also lies in D. Then*

$$\oint_\gamma f(z)\,dz = 0. \tag{2.3.1}$$

Proof. In contrast to Textbook we already made all the necessary calculation in Subsection 1.6.5 "Green's Formula: Complex Variables".

Namely, we proved that

$$\oint_\gamma f(z)\,dz = 2i \iint_Q \frac{\partial f}{\partial \bar{z}}\,dxdy \tag{1.6.7}$$

and since $\dfrac{\partial f}{\partial \bar{z}} = 0$ is our definition of the analyticity, this is 0. □

Remark 2.3.1. We also proved in Subsection 1.6.4 "Green's Formula: Discussion" that

- formula (1.6.7) holds in the case when γ consists of several pieces, provided they are properly oriented
- and if the integrand in the double integral is 0 then the orientation does not matter,
- and if domain D is *simply-connected* (that means, whenever γ is a simple closed curve in D, the inside of γ is also a subset of D then this integral is 0 for any closed piecewise smooth curve γ.

(we did it for Green's formula in real variables, but Green's formula in Complex variables simply inherits these properties).

Still, read Theorem 2 of Section 2.3 in the Textbook (pages 108–109).

2.3.2 Curve deformation

Corollary 2.3.2. *Let f be analytic function in a simply-connected domain D, and let γ and γ' be two curves in D with the same start and end points. Then*

$$\int_\gamma f(z)\,dz = \int_{\gamma'} f(z)\,dz. \tag{2.3.2}$$

Proof. Indeed, $\gamma - \gamma'$ is a closed curve in a simply-connected domain and therefore

$$\int_{\gamma} f(z)\,dz - \int_{\gamma'} f(z)\,dz = \int_{\gamma-\gamma'} f(z)\,dz = 0$$

□

Question. What to do if D is not simply-connected?

Definition 2.3.1. (a) Curve γ' is *deformation* of curve γ if there exists a continuous family of curves γ_s ($s \in [0,1]$), contained in D, with the same start and end points such that $\gamma_0 = \gamma$ and $\gamma_1 = \gamma'$.

(b) Let γ and γ' be two *closed curves* in D. Curve γ' is *deformation* of curve γ if there exists a continuous family of closed curves γ_s ($s \in [0,1]$), contained in D, such that $\gamma_0 = \gamma$ and $\gamma_1 = \gamma'$.

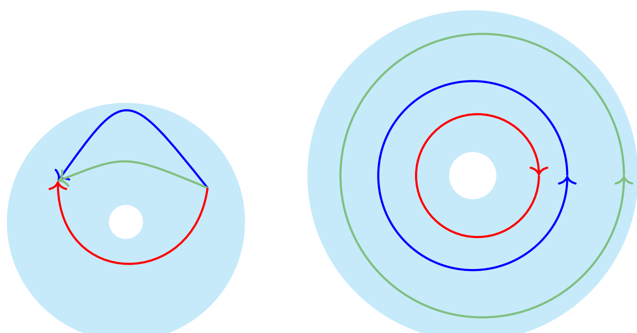


Figure 2.1: Green lines are deformations of blue lines, but red lines are not.

Corollary 2.3.3. *Let f be analytic function in a domain D, and let γ and γ' be either two curves in D with the same start and end points or two closed curves. Let γ' is a deformation of γ. Then* (2.3.2) *holds.*

This corollary allows us to substitute a curve by its deformation.

2.3.3 A primitive of the analytic function

Theorem 2.3.4. *Let f be an analytic function in simply-connected domain D. Then there exists a function F, analytic in D, such that $F'(z) = f(z)$.*

Proof. We define $F(z)$ as

$$F(z) = \int_{z_0}^{z} f(z)\,dz \tag{2.3.3}$$

where the right-hand expression means integral over *any curve γ, with a start-point z_0 (some fixed point in D) and an end-point z.*

This is justified since the right-hand expression *does not depend on the choice of γ with a start-point z_0 (a fixed point in D) and an end-point z.*

Indeed, if γ_1 and γ_2 are tw such points, then

$$\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = \int_{\gamma_1-\gamma_2} f(z)\,dz = 0$$

because $\gamma_1 - \gamma_2$ is a closed curve.

Then

$$dF(z) = \int_{z_0}^{z+dz} f(z)\,dz - \int_{z_0}^{z} f(z)\,dz = \int_{z}^{z+dz} f(z)\,dz = f(z)\,dz$$

and therefore $F'(z) = f(z)$. □

(see also Theorem 3, Section 2.3 of the Textbook, page 109).

Like in Calculus I,

Definition 2.3.2. Function F such that $F' = f$ is called a *primitive of f*.

What happens, if domain is not simply-connected? Then the primitive may be a multiple-valued function.

Example 2.3.1. $f(z) = \frac{1}{z}$ is analytic in $\mathbb{C} \setminus \{0\}$ which is not simply-connected. It's primitive $F(z) = \log(z)$ is multiple-valued.

Sure, if we make a cut, making domain simply-connected (f.e. $(-\infty, 0]$, then we can select a single-valued branch of $\log(z)$.

Theorem 2.3.5. *Let f be an analytic function in domain D. Let F be a primitive of f (if D is not simply-connected, then F may not exist for f, but we assume that it exists). Then* *(i) F is defined up to an additive constant.*

(ii)

$$\int_\gamma f(z)\,dz = F(z_1) - F(z_0), \tag{2.3.4}$$

where γ is any curve with a start-point z_0 and an end-point z_1, so the result is the same for all such curves.

(iii) In particular, this integral is 0 for all closed curves.

Proof. (i) If $F' = f$, then $(F+C)' = F' + C' = f$. on the other hand, if F_1 and F_2 have the same derivative f, then $(F_1 - F_2)' = F_1' - F_2' = 0 \implies F_1 - F_2 = \mathsf{const}$.

(ii) If $f = F'$ then $f\,dz = dF$ and

$$\int_\gamma f\,dz = \int_\gamma dF = F(z_1) - F(z_0)$$

if γ is any curve with a start-point z_0 and an end-point z_1.

(iii) If γ is closed, then $z_1 = z_0$ and the result is 0.

□

Example 2.3.2. (a) If $f(z) = \dfrac{1}{z}$ and γ is a circular curve, centered at 0 with radius r, with counter-clockwise orientation, then

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{dz}{z} = \int_0^{2\pi} i d\theta = 2\pi i \neq 0$$

and therefore a single-valued primitive does not exist if we do not make a cut, preventing from going around 0.

(b) On the other hand, if $f(z) = \dfrac{1}{z^n}$, $n = 2, 3, \ldots$, and γ is a circular curve, centered at 0 with radius r, with counter-clockwise orientation, then

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{dz}{z^n} = \int_0^{2\pi} i e^{-i(n-1)\theta} d\theta = 0.$$

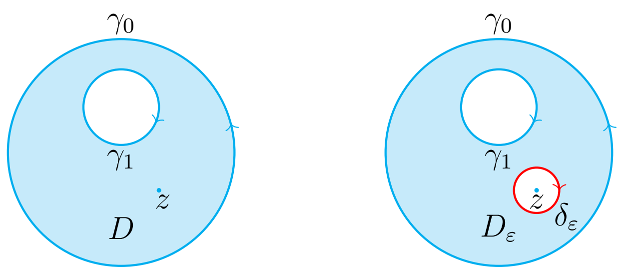
2.3.4 Cauchy's formula

Now we are going to capitalize on Cauchy's theorem and derive out major tool—Cauchy's formula:

Theorem 2.3.6. *Suppose that $f(z)$ is analytic on a domain D and that γ is a piecewise smooth, properly oriented simple closed curve in D whose inside Q also lies in D. Then*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z}. \tag{2.3.5}$$

Proof. First of all recall what is "properly oriented": $\gamma = \gamma_0 + \gamma_1 + \ldots + \gamma_n$, where γ_0 is counter-clockwise and $\gamma_1, \ldots, \gamma_n$ are clockwise oriented. We want



to apply Cauchy's theorem but cannot apply it "out of the box": $\dfrac{f(\zeta)}{\zeta - z}$ has a a singularity at $\zeta = z$. Therefore let us remove from D the disk $D_\varepsilon(z)$ with a boundary γ_ε. The resulting domain $D_\varepsilon = D \setminus D_\varepsilon(z)$ has a boundary $\gamma + \delta_\varepsilon$ where δ_ε is a clockwise oriented circle of radius ε centered at z.

Since now z is not in D_ε, we can apply Cauchy's theorem:

$$\int_{\gamma+\delta_\varepsilon} \frac{f(\zeta)\,d\zeta}{\zeta - z} = 0$$
$$\implies \int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z} = -\int_{\delta_\varepsilon} \frac{f(\zeta)\,d\zeta}{\zeta - z}. \tag{2.3.6}$$

Consider the right-side expression and rewrite it as

$$-\int_{\delta_\varepsilon} \frac{f(z)\,d\zeta}{\zeta - z} - \int_{\delta_\varepsilon} \frac{\big(f(\zeta) - f(z)\big)\,d\zeta}{\zeta - z} \tag{2.3.7}$$

with the first term equal to $2\pi i f(z)$ (indeed, we can move $f(z)$ outside the integral and calculate $\int \frac{d\zeta}{\zeta - z} = -2\pi i$ since δ_ε is clockwise oriented).

Let us *estimate* the second term $\displaystyle\int_{\delta_\varepsilon} \frac{\big(f(\zeta)-f(z)\big)\,d\zeta}{\zeta - z}$ in the right-hand expression of (2.3.7). It does not exceed

$$2\pi\varepsilon \times \frac{1}{\varepsilon}\max_{\zeta\in\delta_\varepsilon}|f(\zeta)-f(z)| = 2\pi\max_{\zeta\in\delta_\varepsilon}|f(\zeta)-f(z)|$$

where $2\pi\varepsilon$ is the length of δ_ε and $|\zeta - z| = \varepsilon$ on δ_ε.

Since f is continuous in z, $\max_{\zeta\in\delta_\varepsilon}|f(\zeta)-f(z)| \to 0$ as $\varepsilon \to 0$.

Therefore, as $\varepsilon \to 0$ the second term in (2.3.7) tends to 0 while the first is $2\pi i f(z)$. Then (2.3.6) implies

$$\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z} = 2\pi i f(z) \implies f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z},$$

which is (2.3.5).

□

Remark 2.3.2. The trick with the removing ε-vicinity of singularity, applying integral theorem, and then tending $\varepsilon \to 0$ and deriving an *integral representation*, is pretty standard in Analysis, and, especially, in PDE. If you take APM346, expect this trick!

So,

$$\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z} = \begin{cases} 2\pi i f(z) & z \text{ inside } \gamma, \\ 0 & z \text{ outside } \gamma, \end{cases} \tag{2.3.8}$$

Indeed, as z outside γ we apply Cauchy's theorem to D without removing $D_\varepsilon(z)$.

What happens if $z \in \gamma$? The integral in the ordinary sense does not exists because integrand is singular as $\zeta = z$. It does not exist also in the sense of ordinary *improper integrals*. Look at

$$\int_{-1}^{1} f(x)\,dx, \qquad f(x) \coloneqq \frac{1}{x}.$$

To calculate it in the sense of improper integrals we use

$$\int_{-1}^{1} f(x)\,dx := \lim_{\varepsilon\to 0^+}\int_{-1}^{-\varepsilon} f(x)\,dx + \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1} f(x)\,dx$$

where *both limits should exist separately.* Instead we introduce

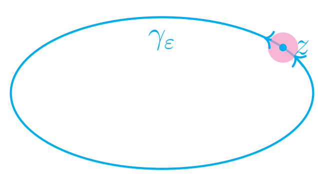
$$v.p.\int_{-1}^{1} f(x)\,dx := \lim_{\varepsilon\to 0^+}\Bigl(\int_{-1}^{-\varepsilon} f(x)\,dx + \int_{\varepsilon}^{1} f(x)\,dx\Bigr).$$

If γ is smooth (does not have an angle) in z, then

$$v.p.\ \int_{\gamma}\frac{f(\zeta)\,d\zeta}{\zeta - z} = \pi i f(z)$$

where $v.p.$ is an abbreviation for "valeur principale" (principal value, fr., more precisely Cauchy principal value) because integral in the sense of "normal" improper integrals does not exist!

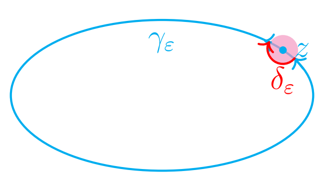
More precisely: we consider small disk $D(z;\varepsilon)$ and denote by γ_ε part of γ outside of $D(z;\varepsilon)$.



Then

$$v.p.\ \int_{\gamma}\frac{f(\zeta)\,d\zeta}{\zeta - z} := \lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon}\frac{f(\zeta)\,d\zeta}{\zeta - z}$$

To prove it we consider domain $D_\varepsilon = D\setminus D(z,\varepsilon)$; its boundary consists of γ_ε and a small arc δ_ε:



Since z is outside D_ε we have

$$\int_{\gamma_\varepsilon} \frac{f(\zeta)\,d\zeta}{\zeta - z} = -\int_{\delta_\varepsilon} \frac{f(\zeta)\,d\zeta}{\zeta - z} = -f(z)\int_{\delta_\varepsilon} \frac{d\zeta}{\zeta - z} - \int_{\delta_\varepsilon} \frac{(f(\zeta) - f(z))\,d\zeta}{\zeta - z}$$

and as $\varepsilon \to 0$ the first integral in the right-hand expression tends to $-\pi i$ and the second to 0.

This notion of v.p. integral is also important in Real Analysis.

Complex variables allow us to calculate many real definite integrals.

Example 2.3.3. Consider

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos(\theta)}$$

with real a, $|a| < 1$ (otherwise denominator vanishes for some θ).

Changing variable $z = e^{i\theta}$, $dz = ie^{i\theta}\,d\theta$ and therefore $\cos(\theta) = \frac{1}{2}(z + \frac{1}{z})$ we get

$$I = \int_\gamma \frac{dz}{iz\bigl(1 + \frac{a}{2}(z + \frac{1}{z})\bigr)} = \int_\gamma \frac{2\,dz}{i\bigl(2z + a(z^2 + 1)\bigr)}$$

with $\gamma = \{z\colon |z| = 1\}$, counter-clockwise oriented. Consider polynomial $P(z) = z^2 + \frac{2}{a}z + 1$; it has two roots, $z_{1,2} = \frac{1}{a}(-1 \pm \sqrt{1 - a^2})$ (with arithmetic root); they are real and $|z_1| < 1$, $|z_2| > 1$.

Example 2.3.1 (continued). Then

$$I = \int_\gamma \frac{2\,dz}{ai(z - z_2)(z - z_1)} = \int_\gamma \frac{f(z)\,dz}{z - z_1}, \qquad f(z) = \frac{2}{ia(z - z_2)},$$

and due to Cauchy formula

$$I = 2\pi i f(z_1) = \frac{4\pi}{a(z_1 - z_2)} = \frac{2\pi}{\sqrt{1 - a^2}}$$

2.4 Consequences of Cauchy's Formula

2.4.1 Analyticity and Power Series

Until the end of Chapter 2 we are going to consider different applications of Cauchy's formula. The first major application is that analytic function can be decomposed into converging power series:

Theorem 2.4.1. *Suppose that $f(z)$ is analytic on a domain D and $z_0 \in D$. Assume that $\{z\colon |z - z_0| < R\} \subset D$.*

Then

(i) f has a power series decomposition

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{2.4.1}$$

converging in this disk.

(ii) Furthermore, the coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}}\,, \tag{2.4.2}$$

where γ is the properly oriented circle $\{z\colon |z - z_0| = r\}$ with any $r\colon 0 < r < R$.

Proof. Let us apply Cauchy's formula

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta\, d\zeta}{\zeta - z}$$

with

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{(\zeta - z_0)} \frac{1}{\left[1 - \dfrac{z - z_0}{\zeta - z_0}\right]}$$

and decompose

$$\frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \Bigl(\frac{z - z_0}{\zeta - z_0}\Bigr)^n$$

which converges as $|z - z_0| < r$ and $|\zeta - z_0| \geq r$ and therefore $\bigl|\dfrac{z - z_0}{\zeta - z_0}\bigr| < 1$.

$$f(z) = \sum_{n=0}^{\infty} \underbrace{\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}}}_{=a_n} \times (z - z_0)^n.$$

This concludes the proof of (2.4.1)–(2.4.2).

For more justification details look at the end of the proof of Theorem 1 of Section 2.4 of the Textbook (pages 124–125). □

Remark 2.4.1. We know already, that the sum of the converging power series is an analytic function. Now we proved the converse statement.

Corollary 2.4.2. *If $f(z)$ is entire analytic function (that means, analytic in $\mathbb{C}$) then it can be decomposed into power series at any point $z_0 \in \mathbb{C}$ and the radius of convergence is $R = \infty$.*

Example 2.4.1.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \qquad \sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!},$$
$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \qquad \sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

have $R = \infty$.

Example 2.4.2. (a)

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad -\operatorname{Log}(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n},$$
$$\frac{1}{(1-z)^m} = \sum_{n=0}^{\infty} \frac{(-1)^{m-1}(m+n-1)! z^n}{n!}$$

with singularity at $z = 1$ have radius of convergence $R = 1$.

(b)

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}, \qquad \operatorname{Arctan}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1}$$

with singularities at $z = \pm i$, have radius of convergence $R = 1$.

Corollary 2.4.3. *Let f be an analytic function in D. Then $f(z)$ is infinitely differentiable in D and, furthermore, its derivatives satisfy*

$$|f^{(n)}(z)| \le C(r) n! r^{-n} \qquad \forall n = 0, 1, \dots \tag{2.4.3}$$

for any $r < \mathsf{dist}(z, \partial D)$, which is the distance from z to ∂D–the boundary of D.

Proof. Immediately from Theorem 2.3.1 and equality $f^{(n)}(z_0) = n!a_n$ (where in the last moment we plug z instead of z_0). Indeed, since the radius of convergence is at least $R = \mathsf{dist}(z, \partial D)$, then $|a_n| \le C(r)r^{-n}$ for $r < R$. □

Remark 2.4.2. (a) If we consider functions of one real variable, then existence of n derivatives does not imply existence of $(n+1)$ derivatives.

(b) $f^{(n)}(x)$ does not necessarily satisfy (2.4.3) or any other similar inequality. In fact, for any sequence b_n there exists an infinitely smooth function $f(x)$ such that $f^{(n)}(x_0) = b_n$.

(c) In this case series $\sum_{n=0}^{\infty} \frac{b_n}{n!}(x - x_0)^n$ does not necessarily converges at all, and even if it converges, the sum is not necessarily $f(x)$.

Theorem 2.4.4 (Fatou's theorem (optional))**.** *Let $f(z)$ be analytic in domain D. Consider power series decomposition at point $z_0 \in D$:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{2.4.1}$$

Assume that the radius of convergence is exactly R and consider circle $\{z\colon |z - z_0| = R\}$. Let Γ be a closed arc of this circle, and also $\Gamma \subset D$. Assume that

$$\lim_{n\to\infty} a_n R^n = 0. \tag{2.4.4}$$

Then (2.4.1) *converges (uniformly) on Γ to $f(z)$.*

We do not give a proof (it is way beyond our reach).

Example 2.4.3 (optional). (a) Consider $-\operatorname{Log}(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$. We know, it converges for $|z| < 1$, and diverges for $z = 1$. What happens, when $|z| = 1$, but $z \ne 1$? Fatou's theorem implies that it converges because condition (2.4.4) is fulfilled.

(b) Consider $\operatorname{Arctan}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1}$. We know, it converges for $|z| < 1$, and diverges for $z = \pm i$. What happens, when $|z| = 1$, but $z \ne \pm i$? Fatou's theorem implies that it converges because condition (2.4.4) is fulfilled.

(c) On the other hand, for $(1-z)^{-1} = \sum_{n=1}^{\infty} z^n$ and $(1+z^2)^{-1} = \sum_{n=0}^{\infty}(-1)^n z^{2n+1}$ condition (2.4.4) fails and these series diverge at any point of $\{z\colon |z|=1\}$.

Example 2.4.4 (optional). Consider

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} a_n z^n, \tag{2.4.5}$$

$$a_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} = \prod_{k=1}^{n}\bigl(1+\frac{\alpha-1}{k}\bigr). \tag{2.4.6}$$

Assuming tht $\alpha \ne 0, -1, -2, \ldots$ (in which case we get a polynomial instead of series) conside

$$\operatorname{Re}(\log(a_n)) = \sum_{k=1}^{n} \operatorname{Re}\Bigl[\log\Bigl(\bigl(1+\frac{\alpha-1}{k}\bigr)\Bigr)\Bigr] \\ \sim \sum_{k=1}^{n} \frac{\operatorname{Re}(\alpha)-1}{k} \sim (\operatorname{Re}(\alpha)-1)\ln(n)$$

(these transitions could be proven easily) and therefore $|a_n| \sim n^{\operatorname{Re}(\alpha)-1}$.

So, $|a_n| \sim n^{\operatorname{Re}(\alpha)-1}$. Then

(a) series (2.4.5)–(2.4.6) is absolutely converging as $|z|=1$ if and only if $\operatorname{Re}(\alpha) < 0$.

(b) for $\operatorname{Re}(\alpha) \ge 1$ it does not converge at any point $z\colon |z|=1$ (common term does not tend to 0)

(c) for $0 \le \operatorname{Re}(\alpha) < 1$ by Fatou's theorem it converges at all points $z\colon |z|=1$, except $z=1$.

Example 2.4.5. Plugging into previous example $z \coloneqq z^2$ and $\alpha = -\frac{1}{2}$ we get that $|a_n| \sim n^{-\frac{1}{2}}$; however, after integration we get

$$\operatorname{Arcsin}(z) = \sum_{n=0}^{\infty} b_n z^{2n+1} \qquad |b_n| \sim n^{-\frac{3}{2}}$$

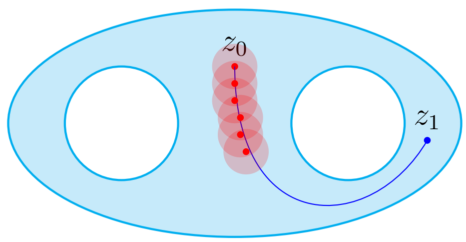
and this series is converging absolutely as $|z|=1$.

2.4.2 Continuation of Analytic Functions

Theorem 2.4.5. *Suppose that $f(z)$ is analytic on a domain D and, further, at some point $z_0 \in D$*

$$f^{(n)}(z_0) = 0, \qquad \forall n = 0, 1, 2, \ldots \tag{2.4.7}$$

Then $f(z) = 0$ in D.



Proof. Consider some other point $z_1 \in D$ and connect z_0 and z_1 by some curve (remember, domain is connected!). Let $\varepsilon > 0$ be the minimal distance between this curve and the boundary of D. Consider disk centered at z_0 of radius δ, then $f = 0$ in it.

Consider another disk of radius δ, centered inside of the first one, down this curve. Then $f = 0$ inside of it due to the same reason, and another, and another ... Eventually we will be able to reach point z_1 and then $f = 0$ in its neighbourhood. □

Corollary 2.4.6. *Let f and g be two analytic functions in domain D, coinciding in non-empty subdomain D'. Then f and g coincide in D.*

Proof. Apply Theorem 2.4.5 to their difference. □

Remark 2.4.3. Again, difference from functions of real variable:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x > 0, \\ 0 & x \le 0, \end{cases}$$

has the property: $f^{(n)}(0) = 0$ for all n, but it is not identically 0.

This method allows sometimes to continue f as analytic function to a larger domain. However for every domain there is an analytic function, which cannot be continued to any larger domain.

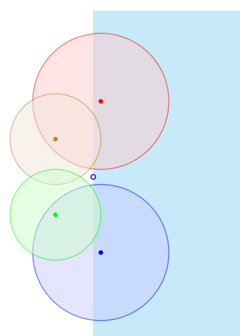
Example 2.4.6. Function

$$f(z) = \sum_{n=0}^{\infty} z^{n!}$$

is analytic in the unit disk, but cannot be continued analytically to any larger domain.

Indeed, as $z = e^{\frac{2\pi pi}{q}}$ for any $q = 1, 2, \ldots$, $p = 0, 1, \ldots, q-1$ this series is *diverging to* ∞ since all terms with $n \geq m$ are just 1. And these points are *dense on the circle* $\{z\colon |z| = 1\}$.

Example 2.4.7. Consider $f(z) = \sqrt{z}$ as $\mathrm{Re}(z) > 0$.



Using the same method we can expand it to the red disk. Then to blue, and to brown, and to green but where brown and green intersect, we got two different branches of $\sqrt{z}$!

2.4.3 The Order of Zero

Suppose that $f(z)$ is analytic and not identically 0 in domain D, and $f(z_0) = 0$ for some $z_0 \in D$. Then the corresponding power series has the first term missing:

$$f(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \ldots, \qquad \text{because} \;\; a_0 = f(z_0) = 0.$$

We know that not all the coefficients here are 0 (otherwise $f(z)$ would be identically 0), so there is an integer m, such that

$$\begin{aligned} & a_0 = a_1 = \ldots = a_{m-1} = 0, \qquad a_m \neq 0 \\ \iff\; & f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \ldots, \qquad a_m \neq 0 \\ \iff\; & f^{(k)}(z_0) = 0 \qquad k = 0, \ldots, m-1, \qquad \text{but} \quad f^{(m)}(z_0) \neq 0. \end{aligned}$$

Definition 2.4.1. (a) In this case we say that z_0 *is a zero of* $f(z)$ and m *is an order of zero.*

(b) *Simple zeroes* are zeroes of order 1, *double zeroes* are zeroes of order 2, *triple zeroes* are zeroes of order 3, and so on.

(c) For convenience we say that z_0 *is zero of order* 0 if it is not a zero at all: $f(z_0) \neq 0$.

Let z_0 be a zero of order m. Then

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-z_0)^m} & z \in D \setminus \{z_0\}, \\ a_m & z = z_0 \end{cases}$$

is an analytic function in D and $g(z_0) = a_m \neq 0$. Indeed,

$$g(z) = a_m + a_{m+1}(z-z_0) + a_{m+2}(z-z_0)^2 + \ldots$$

and one can prove easily that this series is converging in the same disk as power series for $f(z)$.

Theorem 2.4.7. *$f(z)$ has a zero of order m at $z_0 \in D$ if and only if $f(z) = (z-z_0)^m g(z)$ where $g(z)$ is analytic in D and $g(z_0) \neq 0$.*

Example 2.4.8. (a) $\sin(z)$ and $\tan(z)$ have simple zeroes in $\pi n,\ n \in \mathbb{Z}$, $\cos(z)$ and $\cot(z)$ have simple zeroes in $\pi(n+\frac{1}{2}),\ n \in \mathbb{Z}$,

(b) $\cos(z) - 1$ has a double zero at $2\pi n,\ n \in \mathbb{Z}$,

(c) $z\sin(z)$ has a double zero at 0 and simple zeroes at $\pi n,\ n \in \mathbb{Z} \setminus \{0\}$,

(d) $f(z) = \begin{cases} \dfrac{\sin(z)}{z} & z \neq 0, \\ f(0) = 1 \end{cases}$ has simple zeroes in $\pi n,\ n \in \mathbb{Z} \setminus \{0\}$.

2.4.4 Morera's Theorem (optional)

Cauchy's Theorem, on which so much of the development of complex variables depends, has a converse. This theorem, given precisely below, states that if the integral of a continuous function f over every triangle in some domain is zero, then f must be analytic in that domain.

Theorem 2.4.8 (Morera's Theorem)**.** *If f is a continuous function on a domain D and if*

$$\int_\gamma f(z)\,dz = 0$$

for every triangle γ that lies, together with its interior, in D, then f is analytic on in D.

We skip the proof, see proof of Theorem 2 on page 129 of the Textbook.

2.4.5 Liouville's Theorem

Theorem 2.4.9. *Let $F(z)$ be an entire analytic function and let F be bounded, which means that there is a constant M such that $|F(z)| \le M$ for all $z \in \mathbb{C}$. Then $F(z)$ is identically constant.*

Proof. Let $g(z) = \dfrac{F(z) - F(0)}{z}$ (properly defined at $z = 0$); then $g(z)$ is an entire analytic function as well. Then for $z\colon |z| = R > 0$ we estimate

$$|g(z)| \le \frac{|F(z)| + |F(0)|}{R} \le \frac{2M}{R}.$$

For any $\zeta \in \mathbb{C}$ let us take $R > 2|\zeta|$. By Cauchy's formula

$$g(\zeta) = \frac{1}{2\pi i} \int_{|z|=R} \frac{g(z)\,dz}{z - \zeta}.$$

Then

$$|g(\zeta)| \le \frac{1}{2\pi} \frac{2M}{R} \frac{2\pi R}{R - |\zeta|} \to 0 \qquad \text{as} \quad R \to \infty.$$

Then $g(\zeta) = 0$ (identically) and $F(z) = F(0)$. □

Theorem 2.4.10 (optional)**.** *Let $F(z)$ be an entire analytic function and let F satisfy*

$$|F(z)| \le M(|z| + 1)^m \qquad \forall z \in \mathbb{C}.$$

Then $F(z)$ is a polynomial of degree not exceeding m.

Proof. Similarly, consider

$$g(z) = \frac{F(z) - P(z)}{z^{m+1}}, \qquad P(z) = \sum_{k=0}^{m} a_k z^k, \quad a_k = \frac{1}{k!} F^{(k)}(0).$$

Then $g(z)$ is an entire analytic function, and it satisfies

$$|g(z)| \le \frac{|F(z)| + |P(z)|}{|z|^{m+1}} \le \frac{2M(|z|+1)^m}{|z|^{m+1}} \le 4MR^{-1}$$

for $|z| + 1 \le R$ and the rest of the proof remains unchanged. □

2.4.6 Analytic Logarithms

Suppose f is an analytic function in a simply-connected domain D. Assume that f has no zeroes in D.

Fix $z_0 \in D$ and define

$$h(z) = \int_{z_0}^{z} \frac{f'(w)}{f(w)}\, dw$$

with integral taken along any piecewise smooth curve from z_0 to z; since $\frac{f'(w)}{f(w)}$ is an analytic in D (think, why) this integral does not depend on the curve.

Then $h' = \frac{f'}{f}$ and

$$\begin{aligned}(e^{-h(z)} f(z))' =& -h' e^{-h(z)} f(z) + e^{-h(z)} f(z) \\ =& e^{-h(z)} \Big(-f'(z) + f'(z) \Big) = 0.\end{aligned}$$

Then $e^{-h(z)} f(z) = c$ with non-zero constant c and finally $f(z) = ce^{h(z)}$.

Since $h(z_0) = 0$ we have $c = Log(f(z_0))$ and $g(z) = h(z) - \mathrm{Log}(f(z_0))$ we proved

Theorem 2.4.11. *Suppose f is an analytic function in a simply-connected domain D. Assume that f has no zeroes in D. Then there is an analytic function $g(z)$ in D such that*

$$e^{g(z)} = f(z), \qquad z \in D.$$

2.4.7 Multiplication of Power Series (Optional)

We know that the product of two convergent power series is again a power series, and the coefficients of the product series may be obtained from those of the original series by a rather simple rule.

Theorem 2.4.7 of this section allows us to give a simple proof of that result. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

be the two series, each with radius of convergence at least R, $R > 0$.

Then the product $h(z) = f(z)g(z)$ is analytic in the disk $\{z\colon |z| < R\}$. Hence, by Theorem 2.4.1 , h has a power series expansion in this same disk

$$h(z) = \sum_{n=0}^{\infty} c_n z^n \qquad \text{with} \qquad c_n = \frac{h^{(n)}(0)}{n!}.$$

However, the product rule for differentiation gives

$$h^{(n)}(0) = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} f^{(k)}(0) g^{(n-k)}(0)$$

But

$$\frac{f^{(k)}(0)}{k!} = a_k, \qquad \frac{g^{(n-k)}(0)}{k!} = b_{n-k}$$

and therefore

$$c_k = \sum_{k=0}^{n} a_k b_{n-k}.$$

2.5 Isolated Singularities

2.5.1 Classification

In this section we study isolated singularities which will play a crucial role in the rest of the class.

Definition 2.5.1. An analytic function $f(z)$ has an *isolated singularity at a point* z_0 if f is defined and analytic in *the punctured disk* $\{z\colon 0 < |z - z_0| < r\}$, for some $r > 0$.

Three examples of isolated singularities are:

(a) 0 is an isolated singularity for $f_1(z) = \dfrac{z^3}{z}$ and $f_2(z) = \dfrac{\sin(z)}{z}$;

(b) 0 is an isolated singularity for $g_1(z) = \dfrac{2}{z^3}$ and $g_2(z) = \cot(z)$;

(c) 0 is an isolated singularity for $h(z) = \exp\bigl(\dfrac{1}{z}\bigr)$.

As these examples show, there are three possible modes of behaviour for $f(z)$ when $0 < |z - z_0| < r$:

(a) $|f(z)|$ remains bounded as $z \to z_0$; then we call z_0 *a removable singularity.*

(b) $\lim\limits_{z \to z_0} |f(z)| = \infty$; then we call z_0 *a pole.*

(c) Neither of these two cases. Then we call z_0 *an essential singularity.*

As we are going to show,

(a) *a removable singularity* is not really a singularity at all: we can remove it, (re)defining $f(z_0)$ and then $f(z)$ will be analytic in the whole disk $\{z \colon |z - z_0| < r\}$;

(b) *a pole* is in some sense a "regular singularity" (not a standard terminology);

(c) *an essential singularity* is in some sense a "wild singularity" (also not a standard terminology).

Example 2.5.1. The following singularities *are not isolated singularities*:

(a) $z_0 = 0$ and $f(z) = \cot\bigl(\dfrac{1}{z}\bigr)$ because $f(z)$ is singular at points $z_n = \dfrac{1}{\pi n}$ with $n \in \mathbb{Z} \setminus \{0\}$, so there is a sequence of singular points $z_n \to 0$ as $n \to \infty$;

(b) $z_0 = 0$ and $f(z) = \sqrt{z}$, or $\sqrt[3]{z}$, or z^α with $\alpha \notin \mathbb{Z}$ because it cannot be defined as a single-valued analytic function near 0. It is called *a branching point.*

(c) $z_0 = 0$ and $f(z) = \log(z)$ (also a branching point).

We do not cover these points in this section.

2.5.2 Removable singularities

Suppose that $|f(z)| \le M$ in $\{z\colon 0 < |z - z_0| < r\}$. Let

$$g(z) = \begin{cases} f(z)(z-z_0)^2 & 0 < |z - z_0| < r, \\ 0 & z = z_0. \end{cases}$$

This function is analytic in $\{z\colon 0 < |z - z_0| < r\}$ and also is *differentiable at* z_0 because

$$\lim_{z\to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim(z - z_0)f(z) = 0.$$

Then

$$g(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + b_3(z - z_0)^3 + \ldots$$

with $b_0 = g(z_0) = 0$ and $b_1 = g'(z_0) = 0$. Then

$$\begin{aligned} g(z) &= b_2(z - z_0)^2 + b_3(z - z_0)^3 + \ldots \\ \implies f(z) &= b_2 + b_3(z - z_0) + \ldots \end{aligned} \tag{2.5.1}$$

in $\{z\colon 0 < |z - z_0| < r\}$.

Finally, let us set $f(z_0) = b_2$ and then (2.5.1) holds in the whole disk $\{z\colon |z - z_0| < r\}$. Thus $f(z)$ is analytic there. We removed a singularity at z_0 and we call z_0 *a removable singularity*.

Example 2.5.2. (a) Consider $f(z) := \dfrac{z^3}{z} = z^2$ and we set $f(0) = 0$.

(b) Consider

$$f(z) := \frac{\sin(z)}{z} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$$

and we set $f(0) = 1$.

Remark 2.5.1. Compare with Calculus I: if $f(x)$ is bounded it does not mean that it has a limit, and even if it has a limit, it's derivative could be really bad:

Example 2.5.3. 1. $f(x) = \sin\bigl(\dfrac{1}{x}\bigr)$ is bounded but has no limit as $x \to 0$. However in *complex variables* $f(z) = \sin\bigl(\dfrac{1}{x}\bigr)$ has an essential singularity at 0.

2. $f(x) = x\sin(\frac{1}{x})$ is continuous, but $f'(x) = \sin(\frac{1}{x}) - \frac{1}{x}\cos(\frac{1}{x})$ is unbounded. Again, in *complex variables* $f(z) = z\sin\bigl(\dfrac{1}{x}\bigr)$ has an essential singularity at 0.

2.5.3 Poles

Pole are genuine singularities, but we can "tame" them. Assume that

$$\lim_{z\to z_0} |f(z)| = \infty. \tag{2.5.2}$$

Without any loss of the generality we can assume that

$$|f(z)| \geq 1 \qquad \text{for} \quad z\colon |z - z_0| < r. \tag{2.5.3}$$

Indeed, if needed we can reduce r. Then $g(z) = \dfrac{1}{f(z)}$ is analytic and bounded in the punctured disk $\{z\colon 0 < |z - z_0| < r\}$ and therefore for $g(z)$ point z_0 is a removable singularity.

So we can define $g(z_0) = \lim_{z\to z_0} g(z) = 0$ (because of (2.5.2)) and $g(z)$ becomes analytic in the whole disk $\{z\colon |z - z_0| < r\}$.

However, since $g(z_0) = 0$ point z_0 is also a zero of $g(z)$. Let $m = 1, 2, \dots$ be an order of zero. Then

$$g(z) = (z - z_0)^m h(z)$$

where $h(z)$ is analytic in the disk $\{z\colon |z - z_0| < r\}$ and $h(z_0) \neq 0$. Since $g(z)$ does not vanish in the punctured disk $\{z\colon 0 < |z - z_0| < r\}$, neither does $h(z)$, so it does not vanish in the whole disk $\{z\colon |z - z_0| < r\}$.

Therefore $H(z) = \dfrac{1}{h(z)}$ is analytic in $\{z\colon |z - z_0| < r\}$ and $H(z_0) \neq 0$. Then

$$f(z) = \frac{1}{g(z)} = \frac{1}{(z - z_0)^m h(z)} = \frac{H(z)}{(z - z_0)^m}.$$

Since

$$H(z) = b_0 + b_1(z - z_0) + b_2(z - z_0)^2 + \dots, \qquad b_0 \neq 0$$

we conclude that

$$f(z) = a_{-m}(z - z_0)^{-m} + a_{-m+1}(z - z_0)^{-m+1} + a_{-m+2}(z - z_0)^{-m+2} + \dots \tag{2.5.4}$$

with $a_n = b_{n+m}$ and $a_{-m} \neq 0$.

Remark 2.5.2. Decomposition (2.5.4) is an example of what we call later *Laurent's series.*

We see that $f(z)$ has a pole at z_0 iff and only if $g(z) = \dfrac{1}{f(z)}$ (with $g(z_0) = 0$) has a zero at z_0.

Definition 2.5.2. (a) We say, that $f(z)$ has *a pole of order (multiplicity) m at z_0* if $g(z) = \dfrac{1}{f(z)}$ has a zero of order (multiplicity) m at z_0.

(b) Poles of multiplicity 1 are called *simple poles*, poles of multiplicity 2 are called *double poles*, poles of multiplicity 3 are called *triple poles*, and so on.

We the got the following

Theorem 2.5.1. *$f(z)$ has a pole of order m at z_0 if and only if decomposition* (2.5.4) *holds with $a_{-m} \neq 0$ holds in $\{z\colon 0 < |z - z_0 < r\}$.*

Example 2.5.4. (a) $\cot(z)$ has simple poles at $z_n = \pi n$.

(b) $\cot^2(z)$ has double poles at $z_n = \pi n$.

(c) $z\cot^2(z)$ has double poles at $z_n = \pi n$, $n \neq 0$ and a simple pole at $z_0 = 0$.

(d) $z^{-1}\cot(z)$ has simple poles at $z_n = \pi n$, $n \neq 0$ and a double pole at $z_0 = 0$.

2.5.4 Essential singularities

Recall that isolated singularities which are neither removable, nor poles are called *essential singularities*. To show how wild these singularities are we give without proof the following

Theorem 2.5.2 (Great Picard's Theorem)**.** *If an analytic function $f(z)$ has an essential singularity at a point z_0, then on* any *punctured neighbourhood of z_0, $f(z)$ takes all possible complex values, with at most a single exception, infinitely often.*

Instead we prove much weaker

Theorem 2.5.3 (Little Picard's Theorem)**.** *If an analytic function $f(z)$ has an essential singularity at a point z_0, then on* any *punctured neighbourhood of z_0, $f(z)$ approaches all possible complex values.*

Proof. Assume that it does not approach w. Then in $\{z\colon |z - z_0| < r\}$ we have $|f(z) - w| \geq \varepsilon$ for some $\varepsilon > 0$. Then in this disk $g(z) = \dfrac{1}{f(z) - w}$ is an analytic function, and $g(z) = (z - z_0)^m h(z)$ with $m = 0, 1, 2, \ldots$ and analytic $h(z)$, $h(z_0) \neq 0$.

Next,

$$f(z) - w = \frac{1}{g(z)} = (z - z_0)^{-m} H(z)$$

with analytic $H(z)$, $H(z_0) = \dfrac{1}{h(z_0)} \neq 0$. As $m = 0$ we conclude that $f(z)$ has a removable singularity at z_0, and as $m = 1, 2, \dots$ we have a pole. Contradiction. □

Example 2.5.5. (a) $\exp\bigl(\frac{1}{z}\bigr)$ has an essential singularity at 0. It takes any value $w \neq 0$ at points $\dfrac{1}{\operatorname{Log}(w) + 2\pi i n}$, $n \in \mathbb{Z}$.

(b) $\cos\bigl(\frac{1}{z}\bigr)$ has an essential singularity at 0. It takes any value w at points $\dfrac{1}{\pm \operatorname{Arccos}(w) + 2\pi i n}$, $n \in \mathbb{Z}$.

2.5.5 Laurent's Series

We start from Laurent's series, and then cover residues.

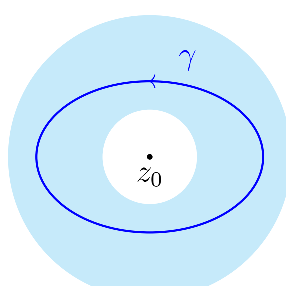
Theorem 2.5.4. *Assume that $f(z)$ is an analytic function in the* annulus *(ring) $D = \{z\colon r < |z - z_0| < R\}$, with $r < R$. Then the following decomposition holds in D:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{2.5.5}$$

with

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)\, dz}{(z - z_0)^{n+1}} \tag{2.5.6}$$

where γ is any closed curve in D, winging exactly once in the counter-clockwise direction around z_0.



Remark 2.5.3. Series

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

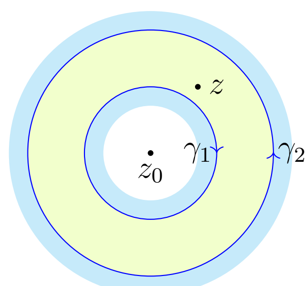
converges in D if and only if *both* series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

converge in D; these series converge in $\{z\colon |z-z_0|<R\}$ and $\{z\colon |z-z_0|>r\}$ correspondingly.

Definition 2.5.3. Series on the right of (2.5.5) is called *Laurent's series.*

Proof of Theorem 2.5.4. Let us consider $z\in D$ and a smaller annulus $D'=\{z\colon r'<|z-z_0|<R'\}$ with $r<r'<R'<R$, containing z, with the border $\gamma_1+\gamma_2$, where $\gamma_2=\{z\colon |z-z_0|=R'\}$, counter-clockwise oriented and $\gamma_1=\{z\colon |z-z_0|=r'\}$, clockwise oriented.



Then according to Cauchy's formula

$$f(z)=\frac{1}{2\pi i}\int_{\gamma_1+\gamma_2}\frac{f(\zeta)\,d\zeta}{\zeta-z}=\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)\,d\zeta}{\zeta-z}+\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)\,d\zeta}{\zeta-z}.$$

Note that $|z-z_0|<|\zeta-z_0|$ for $\zeta\in\gamma_2$ and therefore there

$$\frac{1}{\zeta-z}=\frac{1}{(\zeta-z_0)-(z-z_0)}=\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\zeta-z_0)^{n+1}}$$

exactly like in the proof of Cauchy's formula.

Meanwhile, $|z - z_0| > |\zeta - z_0|$ for $\zeta \in \gamma_1$ and therefore there

$$\frac{1}{\zeta - z} = -\frac{1}{(z - z_0) - (\zeta - z_0)} = -\sum_{n=0}^{\infty} \frac{(\zeta - z_0)^n}{(z - z_0)^{n+1}}.$$

Therefore, plugging into

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)\,d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

we get

$$\begin{aligned} f(z) =& \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}} \times (z - z_0)^n \\ &- \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{\gamma_1} f(\zeta)(\zeta - z_0)^n\,d\zeta \times (z - z_0)^{-n-1}. \end{aligned}$$

Observe that

$$\begin{aligned} &-\sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{\gamma_1} f(\zeta)(\zeta - z_0)^n\,d\zeta \times (z - z_0)^{-n-1} \\ =& \sum_{m=-\infty}^{-1} \frac{1}{2\pi i}\int_{-\gamma_1} \frac{f(\zeta)}{(\zeta - z_0)^{m+1}}\,d\zeta \times (z - z_0)^m \end{aligned}$$

where we changed the sign in front and γ_1 to $-\gamma_1$ and plugged $n = -m - 1$.

Plugging $m := n$ we get

$$\begin{aligned} f(z) =& \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}} \times (z - z_0)^n \\ &+ \sum_{n=-\infty}^{-1} \frac{1}{2\pi i}\int_{-\gamma_1} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}} \times (z - z_0)^n. \end{aligned}$$

We can rewrite it as

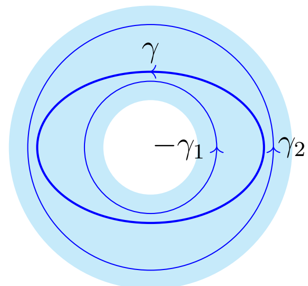
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{2.5.5}$$

with

$$a_n = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}} \tag{2.5.6}$$

because due to Cauchy's theorem integrals over γ, γ_2 and $-\gamma_1$ are equal.

□



2.5.6 Isolated Singularities and Laurent's Series

It is a special case of the previous: simply $r = 0$. So, we have

Corollary 2.5.5. *Let z_0 be an isolated singularity of an analytic function $f(z)$. Then $f(z)$ could be decomposed into Laurent's series converging in the punctured disk $\{z\colon 0 < |z - z_0| < R\}$ for some $R > 0$:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \tag{2.5.5}$$

And the Laurent's series characterize type of the isolated singularity:

Theorem 2.5.6. *Let z_0 be an isolated singularity; consider Laurent's series* (2.5.5). *Then*

(i) z_0 is a removable singularity if and only if $a_n = 0$ for all $n < 0$;

(ii) z_0 is a pole of order $m \geq 1$ if and only if $a_n = 0$ for all $n < -m$ but $a_{-m} \neq 0$;

(iii) z_0 is an essential singularity if and only if there is an infinite number of coefficients with $n < 0$ and $a_n \neq 0$.

Proof. (i) We already have covered removable singularities. Recall that $f(z)$ has a removable singularity at z_0 means that $|f(z) \leq M$ and then the expression for a_n with $n < 0$ tends to 0 as $r \to 0$.

(ii) We also have covered poles when $f(z) \to \infty$ as $z \to z_0$ and proved that $(z - z_0)^m f(z)$ has a removable singularity at z_0 where m is an order of zero of $\dfrac{1}{f(z)}$ at z_0. Conversely, if Laurent's series has a finite number of negative powers then we see that z_0 is a pole.

(iii) So, what is left?—only essential singularities and Laurent's series with an infinite number of negative powers.

□

Example 2.5.6. $z_0 = 0$ is an essential singularity of

(a) $\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n} = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!}$;

(b) $\cos\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!z^{2n}}$;

(c) $\sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!z^{2n+1}}$.

Example 2.5.7. (a) Let z_0 be a pole of $f(z)$. Then z_0 will be a removable singularity for $(z-z_0)^m f(z)$ where m is multiplicity of the pole.

(b) Let z_0 be an essential singularity of $f(z)$. Then z_0 will be an essential singularity for $(z-z_0)^m f(z)$.

Example 2.5.8. (a) $z = 0$ is an essential singularity for $f(z) = \exp(z) + \exp\left(\frac{1}{z}\right)$. It's decomposition is

$$\sum_{n=-\infty}^{0} \frac{z^n}{(-n)!} + \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

(b) $z = 0$ is an essential singularity for $f(z) = \exp(z)\exp\left(\frac{1}{z}\right) = \exp\left(z+\frac{1}{z}\right)$. It's decomposition is

$$\sum_{m=0}^{\infty} \frac{z^m}{m!} \cdot \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{p=-\infty}^{\infty} b_p z^p$$

with

$$b_p = \sum_{n\geq 0, m:\, n-m=p} \frac{1}{n!m!}.$$

2.5.7 Singularities at ∞

Assume that $\infty \in D \subset \widehat{\mathbb{C}}$; that means that ∞ has a neighbourhood contained in D; in other words $D \supset \{z\colon |z| > r\}$ for sufficiently large r. Like in the case $z_0 \in \mathbb{C}$ we have Definitions:

Definition 2.5.4. Let $f(z)$ be an analytic function in $\{z\colon |z| > r\}$. Then ∞ is *an isolated singularity of* $f(z)$.

Definition 2.5.5. (a) ∞ is *a removable singularity* if $|f(z)| \le M$ for $z\colon |z| > r$ with sufficiently large M, r;

(b) ∞ is *a pole* if $\lim_{z\to\infty} |f(z)| = \infty$;

(c) ∞ is *an essential singularity* if neither of these two cases.

Example 2.5.9. The following singularities *are not isolated singularities*:

(a) $z_0 = \infty$ and $f(z) = \cot(z)$ because $\cot(z)$ is singular at points $z_n = \pi n$ with $n \in \mathbb{Z} \setminus \{0\}$, so there is a sequence of singular points $z_n \to z_0$ as $n \to \infty$;

(b) $z_0 = \infty$ and $f(z) = \sqrt{z}$ because it cannot be defined as a single-valued analytic function near ∞. It is called *a branching point*.

We do not cover these points in this section.

Since $\{z\colon |z| > r\}$ is an annulus (with $R = \infty$) we can decompose $f(z)$ here into Laurent's series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n. \tag{2.5.7}$$

On the other hand, if $f(z)$ is analytic as $|z| > r$ then $g(w) := f\left(\frac{1}{w}\right)$ is analytic in $\{w\colon 0 < |w| < r^{-1}\}$ and *type of singularity of $g(w)$ at 0 coincides with a type of singularity of $f(z)$ at ∞.*

Since

$$g(w) = \sum_{n=-\infty}^{\infty} a_{-n} w^n$$

we conclude from Theorem 2.5.6:

Theorem 2.5.7. *Let ∞ be an isolated singularity; consider Laurent's series* (2.5.7). *Then*

(i) ∞ *is a removable singularity if and only if* $a_n = 0$ *for all* $n > 0$*;*

(ii) ∞ *is a pole of order* $m \geq 1$ *if and only if* $a_n = 0$ *for all* $n < m$ *but* $a_m \neq 0$*;*

(iii) ∞ *is an essential singularity if and only if there is an infinite number of coefficients with* $n > 0$ *and* $a_n \neq 0$*.*

Example 2.5.10. (a) Polynomial of degree m has a pole of multiplicity m at ∞.

(b) e^z, $\sin(z)$, $\cos(z)$ have infinite number of positive powers in their decompositions; and we again conclude that they have essential singularity at infinity.

(c) Assume that all singularities of $f(z)$ on $\widehat{\mathbb{C}}$ are isolated and they are poles. Then it can have only a finite number of singularities $z_1, \ldots z_N$, which are poles of multiplicities $m_1, \ldots, m_N$. Then $g(z) = (z - z_1)^{m_1} \cdots (z - z_n)^{m_n} f(z)$ is an entire analytic function and since we assumed that ∞ is also a pole, we have $|g(z) \leq C(|z| + 1)^M$. Then it is a polynomial $P(z)$ and

$$f(z) = \frac{P(z)}{Q(z)} \qquad Q(z) = (z - z_1)^{m_1} \cdots (z - z_n)^{m_n}$$

is *a rational function.*

Conversely, any rational function has all singularities on $\widehat{\mathbb{C}}$ isolated, and they are poles.

2.5.8 The Residue

Now we introduce a very important notion of the residue of analytic function at the isolated singularity. In the next lecture we will prove the Residue Theorem and apply to to many calculations.

Definition 2.5.6. (a) Let $z_0 \in \mathbb{C}$ be an isolated singularity of an analytic function $f(z)$. Decompose $f(z)$ in its neighbourhood (that means, for $z\colon 0 < |z - z_0| < r$ with $r > 0$) into Laurent's series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{2.5.4}$$

Then coefficient a_{-1} is called *the residue of $f(z)$ at z_0* and denoted as $\operatorname{Res}(f; z_0)$:

$$\operatorname{Res}(f; z_0) = a_{-1}. \tag{2.5.8}$$

(b) Let ∞ be an isolated singularity of an analytic function $f(z)$. Decompose $f(z)$ in its neighbourhood (that means, for $z\colon |z - z_0| > r$ with $r > 0$) into Laurent's series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \tag{2.5.4}$$

Then coefficient a_{-1} *with the opposite sign* is called *the residue of $f(z)$ at ∞* and denoted as $\operatorname{Res}(f; \infty)$:

$$\operatorname{Res}(f; \infty) = -a_{-1}. \tag{2.5.9}$$

Remark 2.5.4. (a) Residue at non-isolated singularity *is not defined*.

(b) If $z_0 \neq \infty$ is a removable singularity of $f(z)$, then $\operatorname{Res}(f, z_0) = 0$.

(c) On the contrary, even if ∞ is a removable singularity of $f(z)$, it may happen that $\operatorname{Res}(f, \infty) \neq 0$.

One can ask, why these definitions? Why we select coefficient a_{-1} among others? And why we take an opposite sign at a_{-1} at ∞?

The answer to the first two questions follows from the equality:

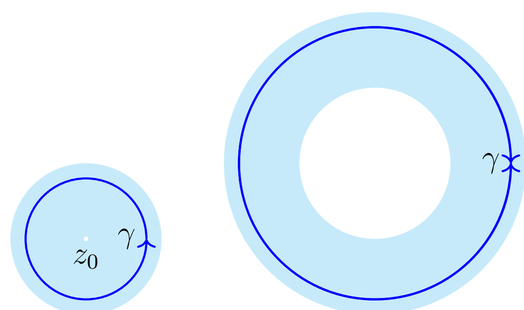
$$\int_\gamma f(z)\, dz = 2\pi i a_{-1} = 2\pi i \operatorname{Res}(f; z_0) \tag{2.5.10}$$

where γ is a circle $\{z\colon |z| = r'\}$ with any sufficiently small r', counter-clockwise oriented. Indeed, it follows from

$$\int_\gamma (z - z_0)^m \, dz = \begin{cases} 2\pi i & m = -1, \\ 0 & m \neq -1. \end{cases} \tag{2.5.11}$$

Exercise 2.5.1. Check it – it is easy!

The answer for the last question follows from the same *first* equality (2.5.10) where γ is a circle $\{z\colon |z| = r'\}$ with any sufficiently large r'.



(a) $z_0 \in \mathbb{C}$ (b) $z_0 = \infty$

However the counter-clockwise oriented circle is going around ∞ *in the wrong direction–a correct direction around* ∞ *is clockwise!* Therefore, to have the left-hand expression in (2.5.10) with *properly (that means, clockwise) oriented* γ, *we need to change the sign in the middle expression*

$$\int_\gamma f(z) \, dz = -2\pi i a_{-1} = 2\pi i \operatorname{Res}(f; \infty). \tag{2.5.12}$$

Therefore we arrive to the universal formula

$$\boxed{\int_\gamma f(z) \, dz = 2\pi i \operatorname{Res}(f; z_0).} \tag{2.5.13}$$

for $z_0 \neq \infty$ and counter-clockwise oriented γ and for $z_0 = \infty$ and clockwise oriented γ.

Remark 2.5.5. Since γ in the definition of $\operatorname{Res}(f; \infty)$ can be any closed curve in the domain $\{z\colon |z| > R\}$ with sufficiently large R, clock-wise oriented and going exactly once around ∞, we conclude that

$$\operatorname{Res}(f(z); \infty) = \operatorname{Res}(f(z + z_0); \infty) \qquad \text{for any} \quad z_0 \in \mathbb{C}. \tag{2.5.14}$$

2.5.9 Computation of Residues

Applications of the Residue Theorem (coming!) requires computation of residues; so there are few recipes how to calculate residues. We consider only $z_0 \in \mathbb{C}$. Recall that then the residue at removable singularity is 0.

Residue at Simple Pole. Let

$$f(z) = \frac{g(z)}{h(z)}, \qquad g(z_0) \neq 0,\ h(z_0) = 0,\ h'(z_0) \neq 0, \tag{2.5.15}$$

and $g(z)$, $h(z)$ analytic at z_0. Then

$$\operatorname{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}. \tag{2.5.16}$$

Indeed, $f(z)$ has a simple pole at z_0 and

$$\begin{aligned} f(z) =& \frac{g(z_0)}{h'(z_0)(z-z_0)} \frac{1 + b_1(z-z_0) + \dots}{1 + c_1(z-z_0) + \dots} \\ =& \frac{g(z_0)}{h'(z_0)(z-z_0)} \Bigl(1 + d_1(z-z_0) + \dots\Bigr) \\ =& \frac{g(z_0)}{h'(z_0)}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + \dots \end{aligned}$$

which implies (2.5.16).

To justify the first equality above observe that

$$F(z) := \frac{1 + b_1(z-z_0) + \dots}{1 + c_1(z-z_0) + \dots}$$

is analytic at z_0 and $F(z_0) = 1$. Here b_k, c_k, d_k and a_k denote some coefficients we do not care about.

Example 2.5.11. (a) $\operatorname{Res}(\cot(z); \pi n) = \operatorname{Res}\bigl(\frac{\cos(z)}{\sin(z)}, \pi n\bigr) = \frac{\cos(z)}{\cos(z)}\Big|_{z=\pi n} = 1.$

(b) $\operatorname{Res}\bigl(\frac{e^{iz}}{z^2+1}; i\bigr) = \frac{e^{iz}}{(2z)}\Big|_{z=i} = \frac{e^{-1}}{2i} = -\frac{i}{2e}.$

(c) $\operatorname{Res}\Bigl(\frac{\cos(z)}{z^2+4}; -2i\Bigr) = \frac{\cos(z)}{(2z)}\Big|_{z=-2i} = \frac{\cosh(2)}{-4i} = \frac{\cosh(2)i}{4}.$

Residue of $(z-z_0)^{-m}H(z)$**.** Let

$$f(z)=\frac{H(z)}{(z-z_0)^m} \tag{2.5.17}$$

with $H(z)$ analytic at z_0 and $m\ge 1$. Then

$$\operatorname{Res}(f;z_0)=\frac{H^{(m-1)}(z_0)}{(m-1)!}. \tag{2.5.18}$$

Indeed,

$$\begin{aligned}H(z)&=\sum_{n=0}^{\infty}b_n(z-z_0)^n,\\ f(z)&=\sum_{n=-m}^{\infty}b_{n+m}(z-z_0)^n\end{aligned}$$

and the coefficient at $(z-z_0)^{-1}$ is b_{m-1}, which implies (2.5.18).

Example 2.5.12. (a) $\operatorname{Res}\Bigl(\frac{\cos(z)}{(z-\pi/4)^3};\frac{\pi}{4}\Bigr)=\frac{-\cos(z)}{2!}\Bigr|_{z=\pi/4}=-\frac{1}{2\sqrt{2}}$. Here $m=3$.

(b) $\operatorname{Res}\Bigl(\frac{\sin(z)}{(z^2+1)^2};i\Bigr)$. Since $m=2$, to apply (2.5.18) we need to have denominator $(z-i)^2$; therefore we set $H(z)=\sin(z)(z+i)^{-2}$. Then

$$\begin{aligned}H'(z)&=\cos(z)(z+i)^{-2}-2\sin(z)(z+i)^{-3}\\ \implies H'(i)&=-\frac{\cosh(1)}{4}-2\frac{i\sinh(1)}{-8i}=-\frac{1}{4e}\end{aligned}$$

and the residue in question is $-\frac{1}{4e}$.

Using Power Series. In more complicated cases of a multiple pole we can use power series.

Hint. (A) It is convenient to move a singularity to 0: $\operatorname{Res}(f;z_0)=\operatorname{Res}(g;0)$ with $g(z)=f(z+z_0)$;

(B) If $f(z)$ is an even function then $\operatorname{Res}(f;0)=0$ (also $\operatorname{Res}(f;\infty)=0$). Indeed, the Laurent's decomposition would contain only even powers of z.

Example 2.5.13. Calculate $\mathrm{Res}(\tan^2(z); z = \pi n + \frac{\pi}{2})$, $n \in \mathbb{Z}$. Using Hint (A)

$$\begin{aligned}\mathrm{Res}(\tan^2(z); z = \pi n + \frac{\pi}{2}) &= \mathrm{Res}(\tan^2(z + \frac{\pi}{2} + \pi n); z = 0) \\ &= \mathrm{Res}(\cot^2(z); z = 0).\end{aligned}$$

Since $\cot(z)$ is odd, $\cot^2(z)$ is even and the required residue is 0 due to Hint (B).

Example 2.5.14. Calculate $\mathrm{Res}(\tan^3(z); z = \pi n + \frac{\pi}{2})$, $n \in \mathbb{Z}$. Using Hint (A),

$$\begin{aligned}\mathrm{Res}(\tan^3(z); z = \pi n + \frac{\pi}{2}) &= \mathrm{Res}(\tan^3(z + \frac{\pi}{2} + \pi n); z = 0) \\ &= \mathrm{Res}(\cot^3(z); z = 0).\end{aligned}$$

Then

$$\cot^3(z) = \frac{\cos^3(z)}{\sin^3(z)} = \frac{\left(1 - \frac{z^2}{2} + \ldots\right)^3}{\left(z - \frac{z^3}{6} + \ldots\right)^3} = \frac{1}{z^3}\frac{\left(1 - \frac{z^2}{2} + \ldots\right)^3}{\left(1 - \frac{z^2}{6} + \ldots\right)^3}$$

where $\ldots$ denote terms which do not contribute to term with z^{-1} in the final decomposition (we need to get no more than z^2 in the numerator and z^3 in denominator); therefore

$$\cot^3(z) = z^{-3}\left(1 - \frac{3z^2}{2} + \ldots\right)\left(1 + \frac{3z^2}{6} + \ldots\right) = z^{-3} - z^{-1} + \ldots$$

and the required residue is -1.

2.6 The Residue Theorem and its Application to the Evaluation of Definite Integrals

2.6.1 The Residue Theorem

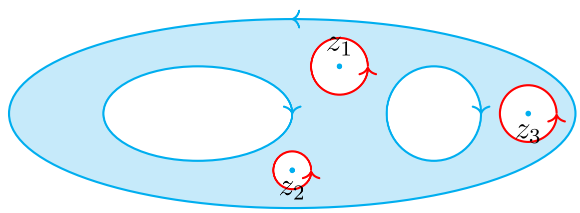
Theorem 2.6.1 (Residue Theorem)**.** *Consider a bounded domain D and its properly oriented boundary γ. Let $f(z)$ be analytic in D with the exception of a finite number of isolated singularities $z_1, \ldots, z_N$, which is also continuous in $\bar{D} = D \cup \gamma$.*

Then

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{N} 2\pi i\,\mathrm{Res}(f; z_k). \tag{2.6.1}$$

Remark 2.6.1. Remember, what is properly oriented? When going along γ, domain is on your left.

Proof. Surround each point z_k by a disk D_k $(k = 1, \ldots, N)$ of a sufficiently small radius, and remove these disks from D; the boundary of the domain $Q = D \setminus (D_1 \cup D_2 \cup \ldots \cup D_N)$ is $\gamma - \gamma_1 - \ldots - \gamma_N$.



Since f is analytic in Q we can apply Cauchy's Theorem:

$$\int_{\gamma-\gamma_1-\ldots-\gamma_N} f(z)\,dz = 0$$

$$\implies \int_\gamma f(z)\,dz = \sum_{k=1}^{N} \int_{\gamma_k} f(z)\,dz$$

due to the properties of line integral.

However, in virtue of the (2.5.1)

$$\int_{\gamma_k} f(z)\,dz = 2\pi i\,\mathrm{Res}(f; z_k), \qquad k = 1, \ldots, N$$

and we arrive to

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{N} 2\pi i\,\mathrm{Res}(f; z_k). \tag{2.6.1}$$

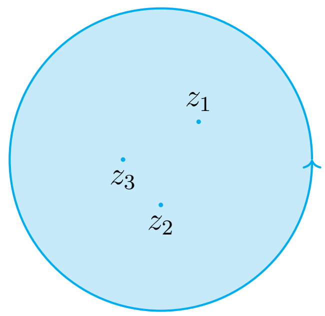
□

Definition 2.6.1. $f(z)$ is a *meromorphic function* if it is analytic on the complex plane $\mathbb{C}$ with the exception of the finite number of isolated singularities at $z_1, \ldots, z_N$.

Corollary 2.6.2. *Let f be a meromorphic function with singularities at $z_1, \ldots, z_N \in \mathbb{C}$. Then*

$$\sum_{k=1}^{N} \operatorname{Res}(f; z_k) + \operatorname{Res}(f; \infty) = 0. \tag{2.6.2}$$

Proof. Consider a domain $D = \{z\colon |z| > R\}$ with a counter-clockwise oriented boundary γ.



According to Theorem 2.6.1

$$\sum_{k=1}^{N} 2\pi i \operatorname{Res}(f; z_k) = \int_\gamma f(z)\, dz = -2\pi i \operatorname{Res}(f; \infty)$$

where the last equality is from the definition of the residue at ∞.

Therefore,

$$\sum_{k=1}^{N} \operatorname{Res}(f; z_k) + \operatorname{Res}(f; \infty) = 0. \tag{2.6.2}$$

□

2.6.2 Integrals of Rational Functions

Consider *a rational function* $f(z) = \dfrac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials of degrees m and n correspondingly. We want to calculate

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx,$$

assuming that

(a) $m \le n-2$,

(b) $Q(x)$ does not have real roots.

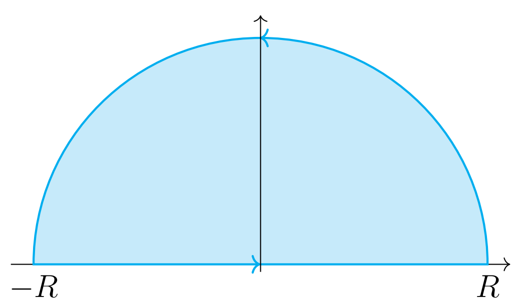
If $m \ge n-1$ then the integral diverges at ∞ (but as $m = n-1$ it could be understood in the sense of the principal value–however we do not cover it). If $Q(x)$ has a real root x_1 then integral diverges at x_1.

Theorem 2.6.3. *Under these assumptions*

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx = 2\pi i \sum_{k=1}^{m} \operatorname{Res}\Bigl(\frac{P(z)}{Q(z)}; z_k\Bigr) \tag{2.6.3}$$

where $z_1, \dots, z_k$ *are all roots of* $Q(z)$ *belonging to* upper complex half-plane $U \coloneqq \{z \in \mathbb{C} \colon\ \operatorname{Im}(z) > 0\}$.

Proof. We want to apply Theorem 2.6.1 and for this we need to take a bounded domain. We take a half-disk $D_R^+ = \{z \colon |z| < R,\ \operatorname{Im}(z) > 0\}$ which contains all roots of $Q(z)$ in the upper half-plane.



Then, according to Theorem 2.6.1, with Γ_R the boundary of D_R^+

$$\int_{\Gamma_R} \frac{P(z)}{Q(z)}\, dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res}\Bigl(\frac{P(z)}{Q(z)}; z_k\Bigr) \tag{2.6.4}$$

where the left part equal to

$$\int_{-R}^{R} \frac{P(x)}{Q(x)}\, dx + \int_{\gamma_R} \frac{P(z)}{Q(z)}\, dz$$

because Γ_R consists of the real segment $[-R, R]$ and an upper arc γ_R.

What happens when $R \to \infty$? Observe that

$$|Q(z)| \geq 2\epsilon_0 |z|^n - M|z|^{n-1} \geq \epsilon_0 |z|^n, \qquad |P(z)| \leq M|z|^m$$

for sufficiently large $|z|$. If you cannot prove it by yourself, look at pages 154–155 of the Textbook.

Therefore, $\left|\frac{P(z)}{Q(z)}\right| \leq M_1 R^{m-n}$ on γ_R and

$$|\int_{\gamma_R} \frac{P(z)}{Q(z)}\, dz| \leq \pi R \times M_1 R^{n-m} \to 0 \qquad \text{as} \quad R \to \infty$$

since $m - n + 1 < 0$. On the other hand,

$$|\frac{P(x)}{Q(x)}| \leq M_2(|x| + 1)^{m-n} \qquad \forall x \in \mathbb{R}.$$

Indeed, it is so for $|x| \geq c$ and for $|x| \leq c$ it follows from the assumption that $Q(x)$ does not have real roots.

Since $\int_{-\infty}^{\infty} M_2(|x|+1)^{m-n}\, dx < \infty$ again because $m - n + 1 < 0$ we conclude that

$$\int_{-R}^{R} \frac{P(x)}{Q(x)}\, dx \to \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx \qquad \text{as} \quad R \to \infty$$

and (2.6.4) implies

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx = 2\pi i \sum_{k=1}^{m} \operatorname{Res}\bigl(\frac{P(z)}{Q(z)}; z_k\bigr). \tag{3.6.3}$$

□

Remark 2.6.2. Using lower half-disk $D' = \{z\colon |z| < R,\ \operatorname{Im}(z) < 0\}$ one could get

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx = -2\pi i \sum_{k=m+1}^{r} \operatorname{Res}\bigl(\frac{P(z)}{Q(z)}; z_k\bigr) \tag{2.6.3$'$}$$

where z_k with $k = m+1, \dots, r$ are roots of $Q(z)$ in the lower half-plane (why minus? – orientation!) but due to Corollary 2.6.2 it is the same since $\operatorname{Res}\bigl(\frac{P}{Q}; \infty\bigr) = 0$ due to $m \leq n - 2$.

Remark 2.6.3. Sure, we can calculate such integrals using decomposition into primitive fractions as in Calculus I.

Example 2.6.1. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+\alpha^2} = \frac{\pi}{\alpha} \qquad \alpha > 0.$

Indeed, there is just one root $z = i\alpha$ in the upper half-plane, and

$$\operatorname{Res}(\frac{1}{z^2+\alpha^2}; \alpha i) = \frac{1}{2z}\Big|_{z=\alpha i} = \frac{1}{2\alpha i}$$

Example 2.6.2. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+\alpha^2)(x^2+\beta^2)} = \frac{\pi}{\alpha\beta(\alpha+\beta)} \qquad \alpha > 0,\ \beta > 0.$

(a) Consider first $\alpha \neq \beta$. Then there are two simple roots $z_1 = \alpha i$ and $z_2 = \beta i$ in the upper half-plane and

$$\operatorname{Res}\Big(\frac{1}{(z^2+\alpha^2)(z^2+\beta^2)}; \alpha i\Big) = \frac{1}{2z(z^2+\beta^2)}\Big|_{z=\alpha i} = \frac{1}{2\alpha(\beta^2-\alpha^2)i}.$$

Similarly the second residue is calculated and their sum is

$$\frac{1}{2\alpha(\beta^2-\alpha^2)i} + \frac{1}{2\beta(\alpha^2-\beta^2)i} = \frac{1}{2\alpha\beta(\alpha+\beta)i}.$$

(b) As $\alpha = \beta$ one can consider a double root, and thus a double pole, but also just plug $\alpha = \beta$ into the result above (not into each residue!).

2.6.3 Integrals over the Real Axis Involving Trigonometric Functions

Theorem 2.6.4. *Let $P(x)$ and $Q(x)$ are even real-valued polynomials of degrees m and n correspondingly, $m \le n-2$ and $Q(x)$ have no real roots. Then*

$$I := \int_{-\infty}^{\infty} \frac{P(x)\cos(x)}{Q(x)}\,dx = \operatorname{Re}\Big[2\pi i \sum_{k=1}^{m} \operatorname{Res}\Big(\frac{P(z)e^{iz}}{Q(z)}; z_k\Big)\Big] \tag{2.6.5}$$

and

$$J := \int_{-\infty}^{\infty} \frac{xP(x)\sin(x)}{Q(x)}\,dx = \operatorname{Im}\Big[2\pi i \sum_{k=1}^{m} \operatorname{Res}\Big(\frac{zP(z)e^{iz}}{Q(z)}; z_k\Big)\Big]. \tag{2.6.6}$$

where $z_1, \dots, z_k$ are all roots of $Q(z)$ belonging to upper complex half-plane $U := \{z \in \mathbb{C}\colon \operatorname{Im}(z) > 0\}$.

Proof. Since $\cos(z)$ and $\sin(z)$ *are unbounded in both the upper (and lower) complex half-plane* observe first that

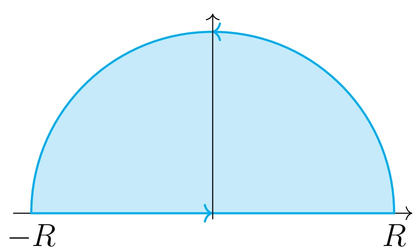
$$I = \operatorname{Re}\Bigl(\int_{-\infty}^{\infty} \frac{P(x)e^{ix}}{Q(x)}\, dx\Bigr) \tag{2.6.7}$$

and

$$J = \operatorname{Im}\Bigl(\int_{-\infty}^{\infty} \frac{xP(x)e^{ix}}{Q(x)}\, dx\Bigr) \tag{2.6.8}$$

and e^{iz} *is bounded in the upper complex half-plane* U.

Then the proof repeats the proof of Theorem 2.6.3. Namely, we consider a half-disk $D_R^+ = \{z\colon |z| < R,\ \operatorname{Im}(z) > 0\}$ and calculate integrals without taking real or imaginary parts using the Residue Theorem.



However, for J in the case of $m = n - 2$ we need some extra arguments. First, we need to prove that J converges (albeit not absolutely). Indeed, integrating by parts we see that

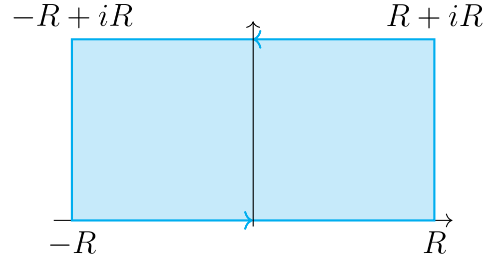
$$\int_1^R f(x)\sin(x)\, dx = -f(x)\cos(x)\Bigr|_{x=1}^{x=R} + \int_1^R f'(x)\cos(x)\, dx,$$

where $f(x) = \frac{xP(x)}{Q(x)} \to 0$ as $r \to +\infty$ and since

$$|f'(x)| \le M(|x|^2+1)^{m-n} \le M(|x|^2+1)^{-2}$$

the integral in the right side converges absolutely. Integral from $-\infty$ to -1 is considered in the same way.

We also need to estimate integral over arc γ_R. The Textbook does it on pages 158–159 (Jordan's Lemma) but we do instead a simpler thing: instead of half-disk we consider a half-square.



[h!]

Then integrals over vertical lines do not exceed

$$M\int_0^R R^{m-n+1}e^{-y}\,dy \le MR^{m-n+1} \le MR^{-1}$$

and integral over upper horizontal line does not exceed

$$M\int_{-R}^R R^{m-n+1}e^{-R}\,dx \le 2MR^{m-n+2}e^{-R} \le 2Me^{-R}$$

and both of those tend to 0 as $R\to\infty$. □

Example 2.6.3. Calculate

$$I=\int_0^\infty \frac{\cos(x)\,dx}{x^2+\alpha^2}, \qquad \alpha>0.$$

Solution. Since the integrand is an even function, we see that we can take integral from $-\infty$ to ∞ and then to *halve it*. Since $Q(x)=x^2+\alpha^2$ has just one root αi in the upper complex half-plane, and this root is simple,

$$I=\frac{1}{2}\operatorname{Re}\Bigl[2\pi i\operatorname{Res}\Bigl(\frac{e^{iz}}{z^2+\alpha^2},\alpha i\Bigr)\Bigr]=\frac{1}{2}\times\operatorname{Re}\Bigl(2\pi i\frac{e^{iz}}{2z}\Bigr|_{z=\alpha i}\Bigr)=\frac{\pi e^{-\alpha}}{2\alpha}.$$

□

Example 2.6.4. Calculate

$$J=\int_0^\infty \frac{x\sin(x)\,dx}{x^2+\alpha^2}, \qquad \alpha>0.$$

Solution. Since the integrand is an even function, we see that we can take integral from $-\infty$ to ∞ and then to *halve it*. Since $Q(x) = x^2 + a^2$ has just one root αi in the upper complex half-plane, and this root is simple,

$$I = \frac{1}{2}\operatorname{Im}\Big[2\pi i \operatorname{Res}\Big(\frac{ze^{iz}}{z^2+\alpha^2}, \alpha i\Big)\Big] = \frac{1}{2} \times \operatorname{Im}\Big(2\pi i \frac{ze^{iz}}{2z}\Big|_{z=\alpha i}\Big) = \frac{\pi e^{-\alpha}}{2}.$$

□

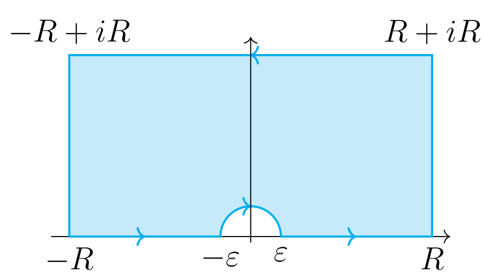
Example 2.6.5. Calculate

$$I := \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx. \tag{2.6.9}$$

Solution. This and the following examples are not following from Theorem 2.6.4 because $Q(x)$ has a real root. We apply the modified method. Observe first that

$$\begin{aligned} I &= \lim_{R\to\infty} \lim_{\varepsilon\to 0^+} \Big[\int_{-R}^{-\varepsilon} \frac{\sin(x)\, dx}{x} + \int_{\varepsilon}^{R} \frac{\sin(x)\, dx}{x}\Big] \\ &= \lim_{R\to\infty} \lim_{\varepsilon\to 0^+} \operatorname{Im} \underbrace{\Big[\int_{-R}^{-\varepsilon} \frac{e^{ix}\, dx}{x} + \int_{\varepsilon}^{R} \frac{e^{ix}\, dx}{x}\Big]}_{=:J_{\varepsilon,R}}. \end{aligned}$$

Now consider a curve $\Gamma_{\varepsilon,R}$:



Due to Cauchy's Theorem $\int_{\Gamma_{\varepsilon,R}} \frac{e^{iz}\, dz}{z} = 0$ (there are no singularities inside). Exactly like in the proof of Theorem 2.6.4 it is easy to show that integrals over both vertical lines and upper horizontal line tend to 0 as $R \to \infty$ and that

$$\int_{1}^{\infty} \frac{e^{ix}\, dx}{x} \qquad \text{and} \qquad \int_{-\infty}^{-1} \frac{e^{ix}\, dx}{x}$$

converge (but not absolutely). Therefore

$$I = -\lim_{\varepsilon\to 0^+} \operatorname{Im}\Bigl(\int_{\gamma_\varepsilon} \frac{e^{iz}\,dz}{z}\Bigr) = -\lim_{\varepsilon\to 0^+} \operatorname{Im}\Bigl(\int_{\gamma_\varepsilon} \frac{dz}{z} + \int_{\gamma_\varepsilon} \frac{(e^{iz}-1)\,dz}{z}\Bigr)$$

where γ_ε is an arc of radius ε.

Observe that

$$\int_{\gamma_\varepsilon} \frac{dz}{z} = \int_\pi^0 i\,d\theta = -\pi i$$

and

$$|\int_{\gamma_\varepsilon} \frac{(e^{iz}-1)\,dz}{z}| \le \pi \max_{\gamma_\varepsilon} |e^{iz}-1| \le M\varepsilon \to 0 \qquad \text{as } \ \varepsilon \to 0^+$$

and therefore $I = \operatorname{Im}(\pi i) = \pi$. □

Example 2.6.6.

$$I := \int_{-\infty}^{\infty} \frac{\sin^2(x)\,dx}{x^2}. \tag{2.6.10}$$

Solution. Observe first that

$$\begin{aligned} I &= \lim_{R\to\infty}\lim_{\varepsilon\to 0^+} \Bigl[\int_{-R}^{-\varepsilon} \frac{(1-\cos(2x))\,dx}{2x^2} + \int_\varepsilon^R \frac{(1-\cos(2x))\,dx}{2x^2}\Bigr] \\ &= \lim_{R\to\infty}\lim_{\varepsilon\to 0^+} \operatorname{Re} \underbrace{\Bigl[\int_{-R}^{-\varepsilon} \frac{(1-e^{2ix})\,dx}{2x^2} + \int_\varepsilon^R \frac{(1-e^{2ix})\,dx}{2x^2}\Bigr]}_{=:J_{\varepsilon,R}}. \end{aligned}$$

Taking the same curve $\Gamma_{\varepsilon,R}$ as in the previous example, we see that due to Cauchy's Theorem $\int_{\Gamma_{\varepsilon,R}} \frac{e^{iz}\,dz}{z} = 0$ (there are no singularities inside).

Again, integrals over both vertical lines and upper horizontal line tend to 0 as $R \to \infty$. Further,

$$\int_1^\infty \frac{(1-e^{2ix})\,dx}{2x^2} \qquad \text{and} \qquad \int_{-\infty}^{-1} \frac{(1-e^{2x})\,dx}{2x^2}$$

converge absolutely. Therefore since $1-e^{2iz} = -2iz + (1+2iz-e^{2iz})$,

$$\begin{aligned} I &= -\lim_{\varepsilon\to 0^+} \operatorname{Re}\Bigl(\int_{\gamma_\varepsilon} \frac{(1-e^{2iz})\,dz}{2z^2}\Bigr) \\ &= -\lim_{\varepsilon\to 0^+} \operatorname{Re}\Bigl(\int_{\gamma_\varepsilon} \frac{-i\,dz}{z} + \int_{\gamma_\varepsilon} \frac{(1-e^{2iz}+2iz)\,dz}{2z^2}\Bigr) \end{aligned}$$

where γ_ε is an arc of radius ε. Here the first integral equals $-\pi$ and the second tends to 0 as $\varepsilon \to 0$ (as in the previous example). Finally, $I = \pi$. $\square$

Remark 2.6.4. To calculate

$$\int_{-\infty}^{\infty} \frac{P(x)\sin(x)\,dx}{xQ(x)} \tag{2.6.11}$$

and

$$\int_{-\infty}^{\infty} \frac{P(x)\sin^2(x)\,dx}{x^2Q(x)} \tag{2.6.12}$$

where $P(x)$ and $Q(x)$ are even real-valued polynomials of degrees m and n correspondingly, $m \le n$, and $Q(x)$ does not have real roots, we use the same curve $\Gamma_{\varepsilon,R}$ and the same approach, albeit instead of Cauchy's theorem we use the Residue theorem.

Example 2.6.7.

$$I := \int \frac{\sin(x)\,dx}{x(x^2+\alpha^2)}, \qquad \alpha > 0, \tag{2.6.13}$$

and

$$J := \int \frac{\sin^2(x)\,dx}{x^2(x^2+\alpha^2)}, \qquad \alpha > 0. \tag{2.6.14}$$

Solution. Analysis of the previous two examples shows that the contribution of 0 (or, more precisely, of $-\gamma_\varepsilon$ with $\varepsilon \to 0^+$) is $\frac{\pi}{\alpha^2}$ (because there is an extra factor $\frac{1}{z^2+\alpha^2}\big|_{z=0} = \frac{1}{\alpha^2}$).

$$\begin{aligned} I = \frac{\pi}{\alpha^2} + \operatorname{Im}\Bigl(2\pi i \operatorname{Res}\Bigl(\frac{e^{iz}}{z(z^2+\alpha^2)}; \alpha i\Bigr)\Bigr) &= \frac{\pi}{\alpha^2} + \operatorname{Im}\Bigl(2\pi i \frac{e^{iz}}{2z^2}\Big|_{z=\alpha i}\Bigr) \\ &= \frac{\pi}{\alpha^2}(1 - e^{-\alpha}) \end{aligned}$$

and

$$\begin{aligned} J = \frac{\pi}{\alpha^2} + \operatorname{Re}\Bigl(2\pi i \operatorname{Res}\Bigl(\frac{1-e^{2iz}}{2z(z^2+\alpha^2)}; \alpha i\Bigr)\Bigr) &= \frac{\pi}{\alpha^2} + \operatorname{Re}\Bigl(2\pi i \frac{(1-e^{2iz})}{4z^3}\Big|_{z=\alpha i}\Bigr) \\ &= \frac{\pi}{\alpha^2}\bigl(1 - \frac{1-e^{-2\alpha}}{2\alpha}\bigr). \end{aligned}$$

$\square$

2.6.4 Integrals Involving $\ln(x)$ or Fractional Powers of x

In this subsection we will deal again mainly with examples to demonstrate approaches than with general theorems. One can formulate theorems easily.

Consider

$$I = \int_0^\infty \frac{x^p P(x)\, dx}{Q(x)} \tag{2.6.15}$$

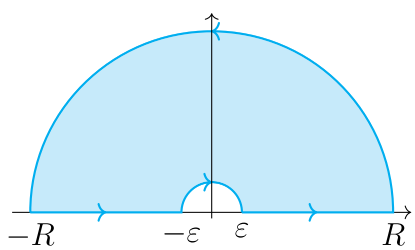
where $P(x)$ and $Q(x)$ are even real-valued polynomials of degrees m and n correspondingly, and $-1 < p < 1$, $p \neq 0$, $m + p + 1 < n$ and $Q(x)$ does not vanish on the real line.

This integral is understood as (a garden variety) *improper integral.*

Example 2.6.8.

$$I = \int_0^\infty \frac{x^p dx}{x^2 + \alpha^2}, \qquad -1 < p < 1, \quad \alpha > 0. \tag{2.6.16}$$

Solution. Consider $f(z) = \dfrac{z^p}{z^2 + \alpha^2}$ and a curve $\Gamma_{\varepsilon,R}$ which is already familiar:



and select a branch z^p which is x^p as $z = x > 0$ and analytic inside $\Gamma_{\varepsilon,R}$.

Then due to the Residue Theorem

$$\int_{\Gamma_{\varepsilon,R}} \frac{z^p\, dz}{z^2 + \alpha^2} = 2\pi i \operatorname{Res}\Bigl(\frac{z^p}{z^2 + \alpha^2}; \alpha i\Bigr) = 2\pi i \frac{(\alpha i)^p}{2z}\Bigr|_{z=\alpha i}$$
$$= \pi \alpha^{p-1} e^{\frac{i\pi p}{2}} \tag{2.6.17}$$

because $i^p = e^{\frac{i\pi p}{2}}$ on *the selected branch.*

Let $I_{\varepsilon,R} = \displaystyle\int_\varepsilon^R \frac{x^p\, dx}{x^2 + \alpha^2}$ denote a required integral, but from ε to R.

On the other hand, integral from $-R$ to $-\varepsilon$ is $e^{i\pi p} I_{\varepsilon,R}$ because $(-1)^p = e^{i\pi p}$ on *the selected branch.* Therefore due to (2.6.17)

$$\bigl(1+e^{i\pi p}\bigr) I_{\varepsilon,R} + \int_{\gamma_R} \ldots + \int_{\gamma_\varepsilon} \ldots = \pi\alpha^{p-1} e^{\frac{i\pi p}{2}} \tag{2.6.18}$$

where γ_R and γ_ε denote large and small arc correspondingly.

Observe that $I_{\varepsilon,R} \to I$ as $\varepsilon \to 0^+$ and $R \to \infty$. Further, $\int_{\gamma_R} \ldots \to 0$ as $R \to \infty$ and $\int_{\gamma_\varepsilon} \to 0$ as $\varepsilon \to 0^+$ (like in the previous lectures).

Then due to (2.6.18)

$$\bigl(1+e^{i\pi p}\bigr) I = \pi\alpha^{p-1} e^{\frac{i\pi p}{2}}$$

and, finally

$$I = \frac{\pi\alpha^{p-1} e^{\frac{i\pi}{2}}}{1+e^{i\pi p}} = \frac{\pi\alpha^{p-1}}{e^{-\frac{i\pi p}{2}} + e^{\frac{i\pi p}{2}}} = \frac{\pi\alpha^{p-1}}{2} \sec\bigl(\frac{p\pi}{2}\bigr).$$

Observe that the final answer is real and positive (as it should be). □

Example 2.6.9.

$$I := \int_0^\infty \frac{\ln(x)\,dx}{x^2+\alpha^2}, \qquad \alpha > 0. \tag{2.6.19}$$

Solution. We repeat arguments of Example 2.6.8, selecting the same domain and curve, and $f(z) = \dfrac{\operatorname{Log}(z)}{z^2+\alpha^2}$ where we select a branch $\operatorname{Log}(z)$ which is $\ln(x)$ as $z = x > 0$ and analytic inside $\Gamma_{\varepsilon,R}$. What will change?

First, we have a different right-hand expression in (2.6.17):

$$2\pi i \operatorname{Res}\Bigl(\frac{\operatorname{Log}(z)}{z^2+\alpha^2}; \alpha i\Bigr) = 2\pi i \frac{\operatorname{Log}(z)}{2z}\Bigr|_{z=\alpha i} = \frac{\pi}{\alpha}\operatorname{Log}(\alpha i) = \frac{\pi}{\alpha}\bigl(\ln(\alpha) + \frac{i\pi}{2}\bigr)$$

because of the branch selection.

Second, while $I_{\varepsilon,R} = \int_\varepsilon^R \frac{\ln(x)\,dx}{x^2+\alpha^2}$,

$$\begin{aligned}\int_{-R}^{-\varepsilon} \frac{\operatorname{Log}(x)\,dx}{x^2+\alpha^2} &= \int_{-R}^{-\varepsilon} \frac{\bigl(\ln(|x|)+\pi i\bigr)\,dx}{x^2+\alpha^2} \\ &= \int_\varepsilon^R \frac{\bigl(\ln(|x|)+\pi i\bigr)\,dx}{x^2+\alpha^2} = I_{\varepsilon,R} + \pi i \int_\varepsilon^R \frac{dx}{x^2+\alpha^2},\end{aligned}$$

where we changed $x \coloneqq -x$ passing from the first line to the second one.

Again, $I_{\varepsilon,R} \to I$ as $\varepsilon \to 0^+$ and $R \to \infty$, $\int_{\gamma_R} \ldots \to 0$ as $R \to \infty$ and $\int_{\gamma_\varepsilon} \ldots \to 0$ as $\varepsilon \to 0^+$ and

$$2I + \pi i \int_0^\infty \frac{dx}{x^2+\alpha^2} = \frac{\pi}{\alpha}\bigl(\ln(\alpha) + \frac{i\pi}{2}\bigr)$$

and taking real part we get $2I = \frac{\pi \ln(\alpha)}{\alpha}$ and $I = \dfrac{\pi\ln(\alpha)}{2\alpha}$ while imaginary part will just bring us trivial $\int_0^\infty \frac{dx}{x^2+\alpha^2} = \frac{\pi}{2\alpha}$. □

The same trick would work in many cases, like with an extra factor x^p with $-1 < p < 1$.

Example 2.6.10.

$$I \coloneqq \int_0^\infty \frac{x\ln(x)\,dx}{(x^2+\alpha^2)(x^2+\beta^2)}, \qquad \alpha > 0,\ \beta > 0,\ \alpha \ne \beta. \tag{2.6.20}$$

Solution. The same trick out of the box would not work here because without $\ln(x)$ we have an odd function and

$$\int_{-R}^{-\varepsilon} \frac{x(\ln(|x|)+\pi i)\,dx}{(x^2+\alpha^2)(x^2+\beta^2)} = -\int_\varepsilon^R \frac{x(\ln(|x|)+\pi i)\,dx}{(x^2+\alpha^2)(x^2+\beta^2)}$$

and adding $\int_\varepsilon^R \frac{x\ln(|x|)\,dx}{(x^2+\alpha^2)(x^2+\beta^2)}$ we get just integral without logarithmic factor.

So, to overcome this problem we add another logarithmic factor and consider $f(z) = \dfrac{z\operatorname{Log}^2(z)}{(z^2+\alpha^2)(z^2+\beta^2)}$.

The "residue answer" will be then

$$\begin{aligned}
&2\pi i \operatorname{Res}\Bigl(\frac{z\operatorname{Log}^2(z)}{(z^2+\alpha^2)(z^2+\beta^2)}, \alpha i\Bigr) + \ldots \\
=&2\pi i \frac{z\operatorname{Log}^2(z)}{2z(z^2+\beta^2)}\Big|_{z=\alpha i} + \ldots \\
=&\pi i \frac{\bigl(\ln(\alpha)+\frac{\pi i}{2}\bigr)^2 - \bigl(\ln(\beta)+\frac{\pi i}{2}\bigr)^2}{\beta^2-\alpha^2} \\
=&\pi i \frac{\bigl(\ln(\alpha)-\ln(\beta)\bigr)\bigl(\ln(\alpha)+\ln(\beta)+\pi i\bigr)}{\alpha^2-\beta^2}.
\end{aligned}$$

On the other hand, the integral expression after we take the limit as $R \to \infty$ and $\varepsilon \to 0^+$ is

$$\begin{aligned}
&\int_0^\infty \frac{x \ln^2(x)\,dx}{(x^2+\alpha^2)(x^2+\beta^2)} + \int_{-\infty}^0 \frac{x\big(\ln(|x|)+i\pi\big)^2\,dx}{(x^2+\alpha^2)(x^2+\beta^2)}\\
=&\int_0^\infty \frac{x \ln^2(x)\,dx}{(x^2+\alpha^2)(x^2+\beta^2)} - \int_0^\infty \frac{x\big(\ln(x)+i\pi\big)^2\,dx}{(x^2+\alpha^2)(x^2+\beta^2)}\\
=&-i\pi\int_0^\infty \frac{x\big(2\ln(x)+i\pi\big)\,dx}{(x^2+\alpha^2)(x^2+\beta^2)}.
\end{aligned}$$

This expression must be equal to the last expression on the previous slide. We take only imaginary parts (taking real parts will bring a much easier result)

$$-2\pi I = \pi\frac{\big(\ln(\alpha)-\ln(\beta)\big)\big(\ln(\alpha)+\ln(\beta)\big)}{\alpha^2-\beta^2}.$$

Taking $\alpha = \beta$ and calculating the limit we would automatically bring result in this case as well:

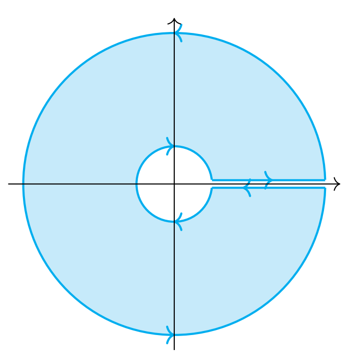
$$-2\pi I = \pi\frac{\ln(\alpha)}{\alpha^2}.$$

□

Example 2.6.11.

$$I := \int_0^\infty \frac{x^p\,dx}{(x-a)^2+\alpha^2}, \qquad -1 < p < 1,\ p \neq 0,\ \alpha > 0. \tag{2.6.21}$$

Solution. If $a \neq 0$, then $(x-a)^2+\alpha^2$ is neither even, nor odd and the previous domain and curve do not work! We need something new, and this is a *keyhole domain.*



Here the radius of the large arc is R, the radius of small arc is ε and the width of the cut is 0!

We take $f(z) = \dfrac{z^p}{(z-a)^2+\alpha^2}$ where z^p is the branch which on the upper side of the cut, when $z = x + i0$, $x > 0$, is equal to x^p. Then on the lower side of the cut, $z = x - i0$, $x > 0$, is equal to $e^{2i\pi p}x^p$.

One can see easily, that the integral over γ_R (the large arc) tends to 0 as $R \to \infty$, integral over γ_ε (the small arc) tends to 0 as $\varepsilon \to 0^+$, and integrals over upper and lower sides of the cut tend to I and $-e^{2i\pi p}I$ correspondingly, because *while we integrate from ε to R on the upper side of the cut, we integrate from R to ε on the lower side!*

So the integral would be

$$(1 - e^{2i\pi p})I$$

in the end.

Now we have not one, but two poles $z_\pm = a \pm \alpha i$. This is why we leave keyhole domain as a last reserve.

Calculating residues

$$\operatorname{Res}\Big(\frac{z^p}{(z-a)^2+\alpha^2}; a \pm \alpha i\Big) = \frac{z^p}{2(z-a)}\Big|_{z=a\pm\alpha i} = \pm\frac{1}{2\alpha i}(a \pm \alpha i)^p.$$

Therefore

$$\begin{aligned}\big(1 - e^{2i\pi p}\big)I =& \frac{\pi}{\alpha}\Big((a+\alpha i)^p - (a-\alpha i)^p\Big)\\ =& \frac{\pi}{\alpha}\Big((a+\alpha i)^p - e^{2i\pi p}(a-\alpha i)^p\Big)\end{aligned}$$

where in the first line ζ^p is a branch defined on $0 < \arg(z) < 2\pi$, while in the second line ζ^p is a branch defined on $-\pi < \arg(z) < \pi$. You just need to look at arguments of ζ and ζ^p.

Then

$$\big(e^{-ip\pi} - e^{i\pi p}\big)I = \frac{\pi}{\alpha}\Big(e^{-ip\pi}(a+\alpha i)^p - e^{i\pi p}(a-\alpha i)^p\Big).$$

Taking imaginary part and observing that in these new settings $\bar{\zeta^p} = \bar{\zeta}^p$ we get

$$-2\sin(\pi p)I = \frac{2\pi}{\alpha}\operatorname{Im}\Big(e^{-i\pi p}(a+\alpha i)^p\Big).$$

Finally,

$$I = -\csc(\pi p)\frac{\pi}{\alpha}\,\mathrm{Im}\Big(e^{-i\pi p}(a+\alpha i)^p\Big).$$

□

Remark 2.6.5. (a) For $a=0$ this answer coincide with the answer obtained in Example 1.

(b) One can consider examples with the factor $x^p \ln^q(x)$.

(c) For

$$I := \int_0^\infty \frac{\ln(x)\,dx}{(x-a)^2+\alpha^2}, \qquad \alpha > 0,$$

one should take a keyhole domain (if $a \ne 0$) and $f(z) = \dfrac{\mathrm{Log}^2(z)}{z^2+\alpha^2}$ with $\mathrm{Log}(z) = \ln(x)$ on the upper side of the cut and $\mathrm{Log}(z) = \ln(x) + 2\pi i$ on it's lower side (think–why square).

(d) With keyhole domain one cannot consider examples with factors $\cos(x)$ or $\sin(x)$ because e^{iz} is bounded only in the upper half-plane.

2.6.5 Integrals of Trigonometric Functions over $[0, 2\pi]$

In this ssubsection we will deal again mainly with examples to demonstrate approaches than with general theorems. One can formulate theorems easily.

Consider

$$I := \int_0^{2\pi} \frac{P\big(\cos(\theta), \sin(\theta)\big)\,d\theta}{Q\big(\cos(\theta), \sin(\theta)\big)} \tag{2.6.22}$$

where $P(x,y)$ and $Q(x,y)$ are real-valued polynomials and $Q\big(\cos(\theta), \sin(\theta)\big) \ne 0$ for all $\theta \in [0, 2\pi]$.

Then one can rewrite it as

$$I := \int_{|z|=1} \frac{P\big(\frac{z}{2}+\frac{1}{2z}, \frac{z}{2i}-\frac{1}{2iz}\big)\,dz}{izQ\big(\frac{z}{2}+\frac{1}{2z}, \frac{z}{2i}-\frac{1}{2iz}\big)} \tag{2.6.23}$$

and use the Residue Theorem.

We start from some old example, we used Cauchy's formula to solve:

Example 2.6.12.

$$I := \int_0^{2\pi} \frac{d\theta}{a - b\cos(\theta)}, \qquad a > |b| > 0. \tag{2.6.24}$$

Solution. Then

$$\begin{aligned} I &= \int_{|z|=1} \frac{dz}{iz\big(a - b(\frac{z}{2} + \frac{1}{2z})\big)} = -\int_{|z|=1} \frac{2i\,dz}{bz^2 + b - 2az} \\ &= -\frac{2}{bi} \int_{|z|=1} \frac{dz}{(z - z_1)(z - z_2)}. \end{aligned}$$

with $z_1 = \frac{1}{b}\big(a - \sqrt{a^2 - b^2}\big)$, $z_2 = \frac{1}{b}\big(a + \sqrt{a^2 - b^2}\big) = z_1^{-1}$, $|z_1| < 1 < |z_2|$. By the Residue Theorem

$$I = -\frac{4\pi}{b} \operatorname{Res}\Big(\frac{1}{(z - z_1)(z - z_2)}; z_1\Big) = -\frac{4\pi}{b} \frac{1}{z_1 - z_2} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

□

Example 2.6.13.

$$I := \int_0^{2\pi} \frac{d\theta}{a^2 - (b + c\cos(\theta))^2}, \qquad a > |b| + |c|,\ |c| > 0. \tag{2.6.25}$$

Solution. Then

$$\begin{aligned} I &= \int_{|z|=1} \frac{dz}{iz\big(a^2 - [b + c(\frac{z}{2} + \frac{1}{2z})]^2\big)} \\ &= \int_{|z|=1} \frac{4iz\,dz}{[2bz + c(z^2 + 1)]^2 - 4a^2z^2}. \end{aligned}$$

The denominator has two roots inside $z_{1,2} = \frac{1}{c}\big(a \pm b - \sqrt{(a \pm b)2 - c^2}\big)$ and two roots outside $z_{3,4} = \frac{1}{z_{1,2}}$. Then

$$I = \frac{4i}{c^2} \int_{|z|=1} \frac{z\,dz}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)},$$

which could be calculated using the Residue Theorem. □

Remark 2.6.6. Example 7 on page 161 of the Textbook is superficial. Indeed

$$\int_0^{2\pi} \frac{d\theta}{a + c\cos^2(\theta)} = 2\int_0^{\pi} \frac{d\theta}{a + c\cos^2(\theta)}$$
$$= \int_0^{\pi} \frac{4d\theta}{2a + c + c\cos(2\theta)} = \int_0^{2\pi} \frac{2\,d\theta}{2a + c + c\cos(\theta)}$$

where in the last equality we made a change $\theta \coloneqq 2\theta$.

And the last integral is exactly what we considered in Example 2.6.12.

Chapter 3

Analytic Functions as Mappings

3.1 The Zeros of an Analytic Function

3.1.1 Introduction

The theme of this chapter is the *geometry of analytic functions*, in particular the nature of their range. This theme manifests itself first in several theorems in Section 1 that pinpoint the number of solutions to an equation of the form $f(z) = w_0$. The answer is formulated in several different ways, all connected closely with the behaviour of the function f on a closed curve.

This ability to count the number of solutions of $f(z) = w_0$ is exploited in Section 2 to prove that the range of a nonconstant analytic function is an open set. This is a profound and elegant property of analytic functions that leads in a natural way to the *maximum modulus principle*.

Even more important, we are led directly to the study of *conformal mapping*, a tool of enormous beauty and power both in applications and theory.

We mainly concentrate on *Möbius map*, also known as *linear fractional transformation.*

3.1.2 Reminder: Definitions

Recall that

(a) z_0 is *a zero of analytic function* f if $f(z_0) = 0$.

(b) If f is not an identical 0, then there exists an integer $m \geq 1$, such that

$$f(z_0) = f'(z_0) = \ldots = f^{(m-1)}(z_0) = 0,\ f^{(m)}(z_0) \neq 0.$$

We call m *the order of zero* z_0.

(c) Equivalently,

$$f(z) = (z - z_0)^m g(z) \qquad \text{with } \ g(z_0) \neq 0 \tag{3.1.1}$$

where $g(z)$ is analytic.

(d) Zero is *simple* if $m = 1$, *double* if $m = 2$, *triple* if $m = 3$ and so on.

3.1.3 Zeroes are Isolated

Theorem 3.1.1. *Suppose that f is analytic in a domain D. Suppose that there are distinct points $z_1, z_2, \ldots \in D$ with $f(z_n) = 0$ for all n and that the sequence $\{z_n\}$ converges to a point $z_0 \in D$.*

Then $f(z) = 0$ for all $z \in D$.

Remark 3.1.1. This theorem states that zeroes of analytic function are *isolated* and cannot *accumulate* to a point in D.

However they can accumulate to the point on the boundary, including ∞. For example, zeroes of $\sin(z)$ are $z_n = \pi n$ and they accumulate to ∞, and zeroes of $\sin\bigl(\frac{1}{z}\bigr)$ which is analytic in $\mathbb{C} \setminus \{0\}$ are accummulating to 0.

Proof of Theorem 3.1.1. Since $f(z_n) = 0$ and $z_n \to z_0 \in D$ we conclude that $f(z_0) = 0$. Let m be an order of zero at z_0, then (3.1.1) holds.

Then $g(z) \neq 0$ for $z\colon |z - z_0| < \varepsilon$ for some $\varepsilon > 0$ and therefore $f(z) \neq 0$ $z\colon 0 < |z - z_0| < \varepsilon$. Contradiction! □

3.1.4 Counting the Number of Zeroes

Now we can think about counting the *number of zeroes* of analytic function $f(z)$ in domain D. However, it is reasonable to *count zeroes together with their orders*, for example, double zero is counted as 2, triple as 3 and so on…

Indeed, counting numbers without their orders is unstable: if $f(z)$ depends continuously on a parameter, then zeroes can merge and split up when this parameter changes, like $z^2+\varepsilon$ has two different simple zeroes $z_{1,2}=\pm\sqrt{-\varepsilon}$ as $\varepsilon\neq 0$, and one double zero as $\varepsilon=0$. More generally, it is the same for roots of polynomials $P(z)$ when coefficients change.

No Gimli's statistics: "That still only counts as one!"

We want also count poles as opposite to zeroes. For example $\dfrac{z-\varepsilon}{z+\varepsilon}$ has a simple pole $z=-\varepsilon$ and a simple zero $z=\varepsilon$ as $\varepsilon\neq 0$ and has none as $\varepsilon=0$ (remember that removable singularity is not a singularity at all!)

Recall that

(a) z_0 is a *pole* of function f analytic in $\{z\colon 0<|z-z_0<\varepsilon\}$, if z_0 is a zero of $h(z)=\dfrac{1}{f(z)}$.

(b) The *order of the pole* n of f at z_0 is the order of zero n of $h(z)$ at z_0; pole is called *simple, double, triple* ... as $n=1,2,3,\dots$ correspondingly.

(c) If z_0 is a pole of order n in z_0 then

$$f(z)=(z-z_0)^{-n}g(z_0) \qquad \text{with} \quad g(z_0)\neq 0 \tag{3.1.2}$$

where again $g(z)$ is analytic (in the neighbourhood of z_0).

Lemma 3.1.2. *(i) Let $f(z)$ have a zero of order m at z_0. Then*

$$\operatorname{Res}\Bigl(\frac{f'}{f};z_0\Bigr)=m. \tag{3.1.3}$$

(ii) Let $f(z)$ have a pole of order n at z_0. Then

$$\operatorname{Res}\Bigl(\frac{f'}{f};z_0\Bigr)=-n. \tag{3.1.4}$$

Remark 3.1.2. (a) If z_0 is a regular point, neither zero nor pole, then $\frac{f'}{f}$ has no singularities at z_0 and the corresponding residue is 0.

(b) If z_0 is an essential singularity, then anything can happen, and we assume that there are no essential singularities in D!

Proof of Lemma 3.1.2. (i) If $f(z)$ has a zero of order m at z_0, then $f(z) = (z-z_0)^m g(z)$,

$$\begin{aligned} f'(z) =& m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z) \\ \Longrightarrow \frac{f'}{f} =& m(z-z_0)^{-1} + \frac{g'}{g} \end{aligned}$$

with $\frac{g'}{g}$ analytic at z_0. Then z_0 is a simple pole of $\frac{f'}{f}$ and $\mathrm{Res}\big(\frac{f'}{f}; z_0\big) = m$.

(ii) If $f(z)$ has a pole of order n at z_0, then $f(z) = (z-z_0)^{-n}g(z)$,

$$\begin{aligned} f'(z) =& -n(z-z_0)^{-n-1}g(z) + (z-z_0)^{-n} g'(z) \\ \Longrightarrow \frac{f'}{f} =& -n(z-z_0)^{-1} + \frac{g'}{g} \end{aligned}$$

with $\frac{g'}{g}$ analytic at z_0. Then z_0 is a simple pole of $\frac{f'}{f}$ and $\mathrm{Res}\big(\frac{f'}{f}; z_0\big) = -n$. □

Now we are able to proof the first major theorem of this Chapter (see Theorem 1 of the textbook on page 173).

Theorem 3.1.3. *Suppose that f is analytic in a domain D except for a finite number of poles.*

Let γ be a piecewise smooth properly oriented simple closed curve in D, which does not pass through any pole or zero of f and whose inside lies in D. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N(f;\gamma) - P(f;\gamma) \tag{3.1.5}$$

where

- *$N(f;\gamma)$ is a number of zeros of f inside γ;*
- *$P(f;\gamma)$ is a number of poles of f inside γ.*

All zeros and poles are counted with multiplicities.

Proof. Let $z_1, \dots, z_p$ be zeroes inside γ and $w_1, \dots, w_q$ be poles inside γ. Then $\dfrac{f'}{f}$ has simple poles at $z_1, \dots, z_p; w_1, \dots, w_q$ and all other points of inside γ are regular.

Due to the Residue Theorem

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz &= \sum_{j=1}^{p} \operatorname{Res}\Bigl(\frac{f'}{f}; z_j\Bigr) + \sum_{k=1}^{q} \operatorname{Res}\Bigl(\frac{f'}{f}; w_k\Bigr)\\ &= \sum_{j=1}^{p} m_j \qquad\qquad\quad + \sum_{k=1}^{q} (-n_k) = N(f;\gamma) - P(f;\gamma)\end{aligned}$$

where equality between the first and the second lines follows from Lemma 3.1.2.

□

3.1.5 Infinity

What about infinity? Recall that

(a) $f(z)$ has a zero of order m at ∞ if $f(z) = z^{-m}g(z)$ with $g(z)$ having removable singularity at ∞ and $g(\infty) \neq 0$. Then

$$\frac{f'}{f} = -mz^{-1} + \frac{g'}{g} \implies \operatorname{Res}\Bigl(\frac{f'}{f}; \infty\Bigr) = m$$

since at ∞ residue is calculated with the opposite sign!

(b) $f(z)$ has a pole of order n at ∞ if $f(z) = z^{n}g(z)$ with $g(z)$ having removable singularity at ∞ and $g(\infty) \neq 0$. Then

$$\frac{f'}{f} = nz^{-1} + \frac{g'}{g} \implies \operatorname{Res}\Bigl(\frac{f'}{f}; \infty\Bigr) = -n$$

since at ∞ residue is calculated with the opposite sign!

Therefore,

Remark 3.1.3. (a) Statement of Lemma 3.1.2 holds for ∞ as well!

(b) Further, Statement of Theorem 3.1.3 holds not only for inside γ but also for outside γ (in which case one must assume that outside γ f is analytic except for a finite number of poles, and ∞ is also either a pole or zero, and it is included then in calculating $N(f;\gamma)$ and $P(f;\gamma)$. Also γ must be properly oriented, so going along it leaves outside on the left!

3.1.6 The Argument Principle

Theorem 3.1.3 states that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N(f;\gamma) - P(f;\gamma) \tag{3.1.5}$$

where γ is a a properly oriented closed curve, and

- $N(f;\gamma)$ is a number of zeros of f inside γ;
- $P(f;\gamma)$ is a number of poles of f inside γ

(and we *always count multiplicities of zeros and poles*).

Observe that $\frac{f'}{f} = \bigl(\log(f)\bigr)'$ and therefore for a simple counter-clockwise and "one-piece" closed curve γ the left-hand expression in (3.1.5) is equal to

$$\begin{aligned}\int_\gamma \frac{f'(z)}{f(z)}\,dz &= \int_\gamma \bigl(\log(f)\bigr)'\,dz\\ &= \log(f(\gamma(t_1)) - \log(f(\gamma(t_0))\end{aligned}$$

where γ is parametrized by $t\in[t_0,t_1]$.

Obviously, the answer is 0 if $\log(f)$ is a single-valued function, which happens, when f has neither 0 nor poles inside γ, but it is not the case otherwise.

In general it is not so, and since $\log(f) = \ln|f| + i\arg(f)$, and $\ln|f|$ is a single valued function, the left-hand expression of (3.1.5) is equal tp

$$i\arg(f(\gamma(t_1)) - i\arg(f(\gamma(t_0))$$

and we arrive to the following Theorem (see Theorem 2 of the Textbook, page 175)

Theorem 3.1.4 (The Argument Principle)**.** *Suppose f is analytic on a domain D except for isolated poles. Let γ be a piecewise smooth positively oriented simple closed curve in D whose inside lies in D and which does not pass through any zeros or poles of f.*

Then,

$$\frac{1}{2\pi}\left[\begin{matrix}\textit{Change of } \arg(f)(z) \\ \textit{as } z \textit{ traverses } \gamma\end{matrix}\right] = N(f;\gamma) - P(f;\gamma) \tag{3.1.6}$$

where γ is a a properly oriented closed curve, and

- *$N(f;\gamma)$ is a number of zeros of f inside γ;*
- *$P(f;\gamma)$ is a number of poles of f inside γ.*

Example 3.1.1. (a) $f(z) = z$ (simple zero at 0) then, when z goes once around 0 and its argument changes from 0 to 2π, $\arg(f(z))$ changes with the same speed from 0 to 2π;

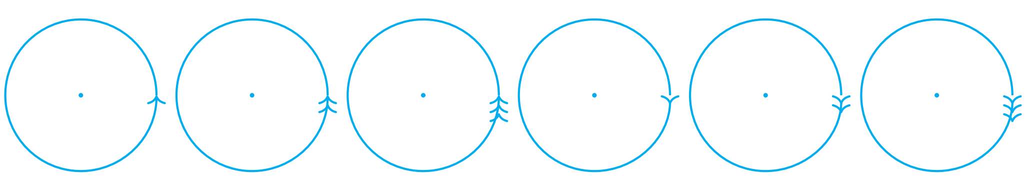
(b) $f(z) = z^2$ (double zero at 0) then, when z goes once around 0 and its argument changes from 0 to 2π, $\arg(f(z))$ changes with the double speed from 0 to 4π;

(c) $f(z) = z^3$ (triple zero at 0) then, when z goes once around 0 and its argument changes from 0 to 2π, $\arg(f(z))$ changes with the triple speed from 0 to 6π;

(d) $f(z) = z^{-1}$ (simple pole at 0) then, when z goes once around 0 and its argument changes from 0 to 2π, $\arg(f(z))$ changes with the same speed but in the opposite direction from 0 to -2π;

(e) $f(z) = z^{-2}$ double pole at 0) then, when z goes once around 0 and its argument changes from 0 to 2π, $\arg(f(z))$ changes with the double speed but in the opposite direction from 0 to -4π;

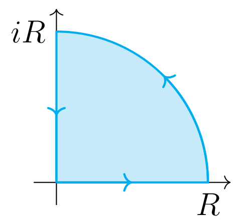
(f) $f(z) = z^{-3}$ triple pole at 0) then, when z goes once around 0 and its argument changes from 0 to 2π, $\arg(f(z))$ changes with the triple speed but in the opposite direction from 0 to -6π.



(a) z (b) z^2 (c) z^3 (d) z^{-1} (e) z^{-2} (f) z^{-3}

Example 3.1.2. Find the number of zeros of the function $f(z) = z^3 - 2z^2 + 4$ in the first quadrant.

Solution. We examine $f(z)$ on the contour shown in figure below, R is very large.



(a) On the horizontal segment, from 0 to R, $z = x$ and $f(x) = x^3 - 2x^2 + 4$ is real and greater than 2. Indeed, $f'(x) = 0 \implies 3x^2 - 4x = 0 \implies x = 0$ or $x = \frac{4}{3}$ with both values ≥ 2 and $f(x) \to +\infty$ as $x \to +\infty$. Therefore its argument stays 0 here.

(b) On the arc $z = Re^{it}$ (with t changing from 0 to $\frac{\pi}{2}$) $f(z) = R^3 e^{3it}\bigl(1 + O(R^{-1})\bigr)$ and therefore argument changes from 0 to $\approx \frac{3\pi}{2}$.

(c) On the vertical segment $z = iy$ (with y changing from R to 0) $f(z) = -y^3 i + 2y^2 + 4$ and it *belongs to the fourth quadrant* $\{w\colon \operatorname{Re}(w) > 0, \operatorname{Im}(w) < 0\}$ and it changes from $\approx \frac{3\pi}{2}$ as $z = Ri$ to 2π as $z = 0$.

Consequently, as z traverses the contour, $\arg(f(z))$ increases by *exactly* 2π, so $f(z) = z^3 - 2z^2 + 4$ has precisely one zero in the first quadrant. □

Remark 3.1.4. (a) In above calculations it is important that $f(z)$ belongs to a specific quadrant (or other part of the plane), which does not allow it to go around 0, so its increment is uniquely defined.

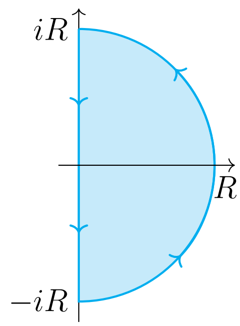
(b) We may calculate $\arg(f(z))$ with a small error, but since its increment is $2\pi n$, $n \in \mathbb{Z}$, the answer is precise.

Example 3.1.3. Let $\lambda \gg 1$. Use the argument principle to show that the equation

$$z + e^{-z} = \lambda$$

has exactly one solution in the right half-plane $\{z : \operatorname{Re}(z) > 0\}$.

Solution. We examine $f(z)$ on the contour shown in figure below, R is very large (depending on λ).



(a) On the vertical segment $z = iy$,

$$h(z) = iy + e^{-iy} - \lambda = (\cos(y) - \lambda) + i(y - \sin(y))$$

has a negative real part (so *it belongs to the left complex half-plane* $\{w\colon \operatorname{Re}(w) < 0\}$); when $y = R$ we have $\arg(h(iy)) \approx \arg(iy) = \frac{\pi}{2}$ and when $y = -R$ we have $\arg(h(iy)) \approx \arg(iy) = \frac{\pi}{2}$.

(b) On the other hand, on the arc, $z = Re^{it}$ and

$$h(Re^{it}) = Re^{it}\big(1 + O(R^{-1})\big);$$

when $t = -\frac{\pi}{2}$ we have $z = -R$ and $\arg(h(z)) \approx \frac{3\pi}{2}$, when t changes slightly, so does $\arg(h(z))$ but then $h(z)$ *stays in the right complex half-plane* until t almost reaches $\frac{\pi}{2}$, and then $\arg(h(R)) \approx \frac{5\pi}{2}$.

Consequently, as z traverses the contour, $\arg(f(z))$ increases by *exactly* 2π, so $h(z) = z - e^{-z} - \lambda$ has precisely one zero in the first quadrant. □

3.1.7 Rouché's Theorem

Theorem 3.1.5 (Rouché's Theorem)**.** *Suppose f and g are analytic on an open set containing an piecewise smooth simple closed curve γ and its inside. Also suppose that*

$$|f(z) + g(z)| < |f(z)| \qquad \forall z \in \gamma. \tag{3.1.7}$$

Then f and g have an equal number of zeros inside γ, counting multiplicities.

Proof. Note that the assumption (3.1.7) (*strict inequality!*) ensures that neither f nor g is zero on γ. We may assume further that all the common zeros of f and g inside γ have been canceled; this affects neither the assumptions nor the conclusion.

Let $h = \frac{g}{f}$; then

$$|h(z) + 1| < 1 \qquad \forall z \in \gamma$$

so the range of h on γ lies in the open disk of radius 1 centered at the point -1.

In particular, $\arg(h(z))$ has no net change as z traverses γ because $w = f(z)$ cannot go around 0. By the Argument Principle, the number of zeros of h inside γ equals the number of poles of h inside γ.

But the number of zeros of h is just the number of zeros of g, and the number of poles of h just the number of zeros of f. □

Remark 3.1.5. (a) Suppose that f and g satisfy the hypotheses of Theorem 3.1.5 except that

$$|f(z) - g(z)| < |f(z)| \qquad \forall z \in \gamma. \tag{3.1.8}$$

It is then still true that f and g have equally many zeros inside γ. The reasoning is elementary: according to Theorem 3.1.5, the assumption (3.1.8) implies that f and $-g$ have equally many zeros inside γ.

However, a point is a zero of $-g$ exactly when it is a zero of g.

(b) So, we need to construct a *a reference function* $g(z)$, satisfying (3.1.7) or (3.1.8) and such that we can easily calculate the number of its zeros.

Example 3.1.4. Show that all the zeros of

$$p(z) = 3z^3 - 2z^2 + 2iz - 8$$

lie in the annulus $\{z\colon 1 < |z| < 2\}$.

Solution. (a) On the circle $\{z\colon |z| = 1\}$

$$|f(z) + 8| \leq 3 + 2 + 2 = 7 < 8,$$

so $p(z)$ and $f(z) = 8$ have the same number of zeros within $\{z\colon |z| = 1\}$, by Rouché's Theorem; thus, $p(z)$ does not vanish in the disk $\{z\colon |z| = 1\}$.

(b) Furthermore, on $\{z\colon |z| = 2\}$,

$$|p(z) - 3z^3| \le 20 < 24 = |3z^3|$$

so $p(z)$ and $g(z) = 3z^3$ have an equal number of zeros within the circle $\{z\colon |z| = 2\}$; since $g(z)$ has 3 zeros here (more precisely, one triple zero $z = 0$), we conclude that $p(z)$ has three of its zeros within $\{z\colon |z| = 2\}$.

Consequently, all the zeros of $p(z)$ lie in the annulus $\{z\colon 1 < |z| < 2\}$.

□

Example 3.1.5. Consider

$$P(z) = z^3 + 2z - 3 - i$$

and, using the argument theorem and Rouchè's theorem calculate the number of its roots in each of the following domains:

(a) $\{z\colon |z-1| < 1\}$;

(b) $\{z\colon |z-1| > 1, |z| < 2\}$,

(c) $\{z\colon |z| > 2\}$.

Solution. (a) As $|z| = 2$ consider $Q(z) = z^3$; then $|Q(z)| = 8$,

$$|P(z) - Q(z)| = |2z - 3 - i| \le 4 + |3 + i| = 4 + \sqrt{10} < 8;$$

therefore $P(z)$ has as many roots in $\{z\colon |z| < 2\}$ as $Q(z)$ has, which is 3 (we count orders).

(b) Then there are no roots in $\{z\colon |z| > 2\}$ due to Principal Theorem of Algebra.

(c) Consider $z = w + 1$ with $|w| = 1$; then

$$P(w+1) = w^3 + 3w^2 + 5w - i$$

and we set $Q(w) = 5w$; then $|Q(w)| = 5$,

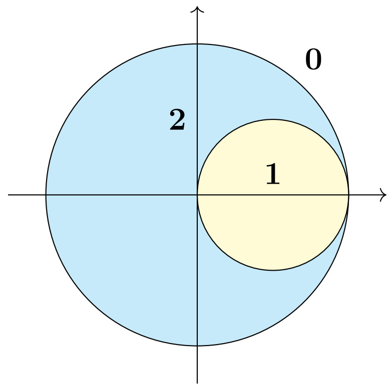
$$|P(w) - Q(w)| = |w^3 + 3w^2 - i| < 5.$$

Indeed, $|P(w) - Q(w)| \le |w + 3| + 1 < 5$ except $w = 1$, and for $w = 1$ we have $|w^3 + 3w^2 - i| = |4 - i| = \sqrt{17} < 5$.

Therefore $P(w+1)$ has as many roots in $\{w\colon |w| < 1\}$ as $Q(w)$ has, which is 1 (we count orders).

Finally, in the domain $\{z\colon |z-1| > 1, |z| < 2\}$, there are $3 - 1 = 2$ roots.

□



Example 3.1.6. Show that the equation

$$e^z = e^2 z$$

has a real root in the unit disk $\{z\colon |z| < 1\}$.

Are there non-real roots?

Solution. We are looking for zeros of $f(z) = e^2 z - e^z$. Compare it with $g(z) = e^2 z$. On the boundary $|g(z)| = e^2$ and

$$|g(z) - f(z)| = |e^z| \le e^{\mathrm{Re}(z)} \le e < e^2.$$

Therefore $f(z)$ has as many roots in the unit disk as $g(z)$, which has one (at 1).

So, $f(z)$ has exactly 1 root there: $e^z = e^2 z \implies \bar{e^z} = \bar{e^2 z}$, $e^{\bar{z}} = e^2 \bar{z}$ and therefore $\bar{z}$ is also a root; since there is just one root, $\bar{z} = z$ and z is real.

Alternatively, consider a real function $f(x) = e^x - e^2 x$, $x \in (0, 1)$. Then $f(0) = 1 > 0$ and $f(1) = e - e^2 < 0$ and since f is a continuous function, there is $x \in (0, 1)$ s.t. $f(x) = 0$. $\square$

Corollary 3.1.6. *(i) Assume that $f(z) = \frac{G(z)}{H(z)}$ where G and H are polynomials. Then*

$$N(f) = P(f) \tag{3.1.9}$$

where $N(f)$ and $P(f)$ are number of zeroes and poles of f on the extended complex plane $\widehat{\mathbb{C}}$ (including ∞ and counting orders).

(ii) In particular, if $f(z) = G(z)$ is a polynomial, then the number of roots, counting multiplicities is equal to the order of polynomial G.

Proof. Indeed, G has just one pole, namely, ∞ and the order of it is equal to the order of polynomial G. $\square$

Remark 3.1.6. The last statement is known as a *Fundamental Theorem of Algebra* (of polynomials).

3.2 Maximum Modulus and Mean Value

3.2.1 Maximum Modulus

Suppose that $f(z)$ is an analytic function on a domain D. We saw in the beginning of Chapter 2 that if the range of f lies in a circle or on a straight line, then f is necessarily constant. We shall see now that a great deal more is true: *either f is constant on D or the range of f is an open set.*

Suppose that f is not identically constant, and let $w_0 = f(z_0)$ be an arbitrary point in the range of f. Then since f is not identically constant, function $f(z) - w_0$ is not identically zero, and so $f(z) - w_0$ has a zero of order $m \geq 1$ at z_0.

Choose $r > 0$ so small that $f(z) - w_0$ has no zero in the punctured disk $\{z\colon 0 < |z - z_0| \leq r\}$; this is possible because the zeros of a non-constant analytic function are isolated (see Section 1).

Let $\delta > 0$ be the minimum value of $|f(z) - w_0|$ for all $z\colon |z - z_0| = r$, and let w be any point with $|w - w_0| < \delta$. Then on the circle $\{z\colon |z - z_0| = r\}$,

$$|(f(z) - w) - (f(z) - w_0)| = |w - w_0| < \delta \leq |f(z) - w_0|. \tag{3.2.1}$$

Therefore, Rouché's Theorem implies that the two functions $f(z) - w$ and $f(z) - w_0$ have an equal number of zeros in the disk $\{z\colon |z - z_0| \leq r\}$.

However, $f(z) - w_0$ has exactly m zeros, and hence so does $f(z) - w$. This shows that *each point w_0 in the range of f lies at the center of a small disk, which is also within the range of f*. Thus, we have established the following:

Theorem 3.2.1 (Theorem 1 in the Section 3.2 of the Textbook)**.** *Suppose that $f(z)$ is a non-constant analytic function on a domain D. Then the range of $f(z)$, as z varies over D, is an open set.*

We can draw an immediate conclusion from the work that preceded the statement of Theorem 3.2.1.

Theorem 3.2.2. *Suppose that $f(z)$ is a non-constant analytic function on a domain D and that $f(z) - f(z_0)$ has a zero of order m at z_0.*

Then f is m-to-1 near z_0, that means that preimage of $w \neq w_0$ contains m distinct points (if we look only at points within the disk $\{z\colon |z - z_0| < r\}$; in particular, if $f'(z_0) = 0$, then f is not one-to-one in any disk containing z_0).

Theorem 3.2.3 (The Maximum-Modulus Principle)**.** [1] *If $f(z)$ is a non-constant analytic function on a domain D, then $|f(z)|$ can have no local maximum on D.*

Proof. Assume that $z_0 \in D$ is a point of local maximum: $|f(z_0)| \geq |f(z)|$ for all $z\colon |z - z_0| < r$.

Then $f(z_0)$ lies on the boundary of the open set $W := \{f(z)\colon |z - z_0| < r\}$. This is in contradiction to the fact that W is an open set containing $f(z_0)$. □

The maximum-modulus principle (Theorem 3.2.3) has ramifications for the real and imaginary parts of an analytic function. Suppose that f is a non-constant analytic function on a domain D; let $u = \operatorname{Re}(f)$ and $g = e^f$. Then both g and $\frac{1}{g}$ are analytic on D, so neither of the functions $|g| = e^u$ and $|\frac{1}{g}| = e^{-u}$ has a local maximum in D.

Therefore, u has no local maxima and no local minima in D: Further, if we apply the same arguments to $if(z)$ instead of $f(z)$ then we would have the same conclusion for $v = -\operatorname{Im}(f)$ and we arrive to the following

Theorem 3.2.4. *Let f is analytic and non-constant on a domain D.*

(i) $\operatorname{Re}(f)$ *has no local maxima and no local minima on D.*

(ii) $\operatorname{Im}(f)$ *has no local maxima and no local minima on D.*

Remark 3.2.1. This is a special case of more general theorem, proven in PDE: Let u be a harmonic function in domain $D \subset \mathbb{R}^n$. Then, either u is constant in D or f has no local maxima and no local minima in D.

Let us continue with maximum-modulus principle a bit further. Let D be a bounded domain and let B be the boundary of D. Then *the closure* of D, $D \cup B$ is both closed and bounded. We know from Calculus II that a

[1] Corollary 1 of theTextbook

continuous real-valued function on a *compact set* (that means closed and bounded set) actually attains its maximum value.

Let us apply this to each of the three functions $|f|$, $\operatorname{Re}(f)$, and $-\operatorname{Re}(f)$, where f itself is analytic on D and continuous on $D \cup B$. We know that each of these three functions is continuous on $D \cup B$ and so must attain its maximum somewhere on $D \cup B$.

If f is non-constant, then the maximum can not be attained on D itself. Hence, the maximum must be attained on B, the boundary of D. Quite obviously, the same conclusion holds if, in fact, f is constant on D. Thus, in all cases, $|f|$, $\operatorname{Re}(f)$, and $-\operatorname{Re}(f)$ attain their maximum values on B, the boundary of D:

Corollary 3.2.5. *If f is analytic on a bounded domain D and continuous on $D \cup B$, where B is the boundary of D, then each of $|f|$, $\operatorname{Re}(f)$, and $-\operatorname{Re}(f)$ attains its maximum value on B (and it is also true for $\operatorname{Im}(f)$ and $-\operatorname{Im}(f)$).*

Corollary 3.2.6. *Let $\operatorname{Re}(f) = 0$ everywhere on B. Then $\operatorname{Re}(f) = 0$ in D, so f itself is constant on all of D.*

Remark 3.2.2. (a) This is a very special case of the uniqueness theorem in PDE: let u be a harmonic function in the bounded domain $D \subset \mathbb{R}^n$, and $u = 0$ on its boundary. Then $u = 0$ in D. Moreover, if u and v are two harmonic functions in D; then $u - v$ is also harmonic in D and if $u = v$ everywhere on B, then $u = v$ everywhere in D.

(b) Assumption that D is being bounded is not a superfluous here. For instance, $f(z) = iz$ is analytic and non-constant on the whole plane, $\operatorname{Re}(f) = 0$ on the real axis, and yet f is not constant on the upper half-plane $\{z\colon \operatorname{Im}(z) > 0\}$, whose boundary is the real axis.

Example 3.2.1.

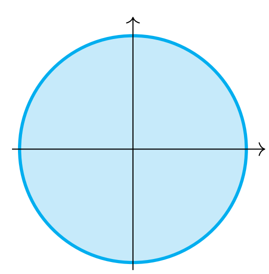
Let u be a harmonic function in the disk $D = \{(x, y)\colon x^2 + y^2 \le 9\}$, equal to

$$g(\theta) = \frac{|\sin(\theta)|}{2 + \cos(\theta)}$$

on its boundary, where θ is a polar angle. Find

$$\max_D u \qquad \text{and} \qquad \min_D u$$

and the points, where they are achieved.



Solution. Since g (and thus u also) is symmetric with respect to x we can consider only points with $\theta\colon 0 \le \theta \le \pi$.

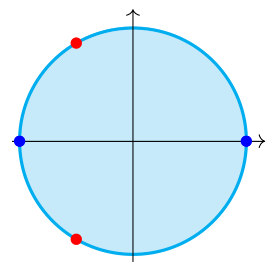
Finding maxima and minima of g: on the upper half-circle with $0 \le \theta \le \pi$

$$g(\theta) = \frac{\sin(\theta)}{2+\cos(\theta)} \implies g'(\theta) = \frac{2\cos(\theta)+1}{(2+\cos(\theta))^2} = 0$$
$$\implies \cos(\theta) = -\frac{1}{2} \implies \theta = \frac{2\pi}{3} \implies g(\theta) = \frac{1}{\sqrt{3}}.$$

However we need to explore the ends $\theta = 0$ and $\theta = \pi$, where $g(\theta) = 0$. So

(a) $\max_D(u) = \dfrac{1}{\sqrt{3}}$ achieved as $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$ (don't forget about lower half circle), or points $-\frac{3}{2} \pm i\frac{3\sqrt{3}}{2}$.

(b) $\min_D(u) = 0$ achieved as $\theta = 0$ and $\theta = \pi$, or points ± 3.



□

Example 3.2.2.

Let

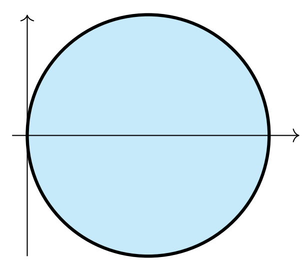
$$f(z) = z^2$$

and

$$D = \{z\colon |z-1| \le 1\}.$$

Find

$$\max_{z\in D} \operatorname{Im}(f(z)) \qquad \text{and} \qquad \min_{z\in D} \operatorname{Im}(f(z))$$



and point(s), where it is achieved.

Solution. Since maxima and minima are achieved on the boundary, we consider $z = a + \cos(t) + i\sin(t)$ with $a = 1$. Then

$$\operatorname{Im} f(z) = 2\big(a+\cos(t)\big)\sin(t) =: 2\phi(t)$$

and

$$\phi'(t) = a\cos(t) + \cos^2(t) - \sin^2(t) = 2\cos^2(t) + a\cos(t) - 1 = 0$$
$$\implies \cos(t) = \frac{1}{4}\Big(-a \pm \sqrt{a^2+8}\Big).$$

We get two values

$$\cos(t_1) = \frac{1}{2} \implies x_{1,2} = \frac{3}{2},\ y_{1,2} = \pm\frac{\sqrt{3}}{2} \implies \operatorname{Im}(z_{1,2}) = \pm\frac{\sqrt{3}}{2},$$

and

$$\cos(t_3) = -1 \implies x_3 = 0,\ y_3 = 0 \implies \operatorname{Im}(z_3) = 0.$$

Therefore, $\max_D f(z) = \frac{\sqrt{3}}{2}$, achieved at $\frac{1}{2}\Big(3+\sqrt{3}i\Big)$ and $\min_D f(z) = -\frac{\sqrt{3}}{2}$, achieved at $\frac{1}{2}\Big(3-\sqrt{3}i\Big)$. □

Example 3.2.3.

Let

$$f(z) = ze^{-z}$$

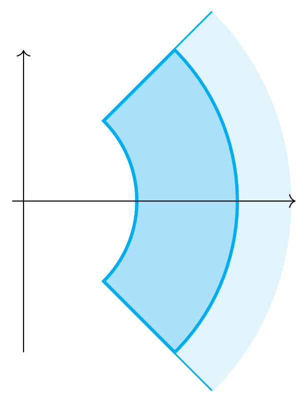
and

$$D = \{z\colon |\operatorname{Im}(z)| < \operatorname{Re}(z),\ |z| > 1\}.$$

Find

$$\max_{z\in D} |f(z)|$$

and point(s), where it is achieved.

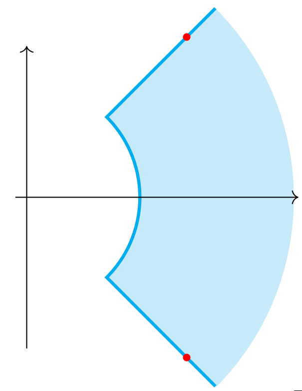


Solution. We cannot apply maximum modulus principle *directly* to the unbounded domain D, therefore we consider *a truncated domain* $D_R := \{z \in D\colon |z| < R\}$, observing that $f(z) \to 0$ as $|z| \to \infty$ while $z \in D$. Therefore $\max_{z\in D} |f(z)|$ is achieved on the boundary of D.

Consider boundary of D, consisting of two rays $\{z\colon z = (1\pm i)t,\ t > \frac{1}{\sqrt{2}}\}$ and an arc $\{z\colon z = e^{i\theta}, -\frac{\pi}{4} < \theta < \frac{\pi}{4}\}$.

Then on the rays $f(z) = (1\pm i)te^{-t\mp it}$ and $|f(z)| = \phi(t) := \sqrt{2}te^{-t}$ and $\max_{t\ge 1/\sqrt{2}} \phi(t) = \sqrt{2}e^{-1}$ achieved as $t = 1$ (one can check that $\phi'(t) > 0$ in the corners $t = \frac{1}{\sqrt{2}}$ and they are not the points of the maximum).

Meanwhile, on the arc $|f(z)| = e^{-\cos(\theta)}$ and the maximum achieved in the corners.



Therefore

$$\max_{z\in D} |f(z)| = \sqrt{2}e^{-1}$$

achieved as $z = (1 \pm i)$.

□

Example 3.2.4.

Let

$$f(z) = \frac{1}{1+z^2}$$

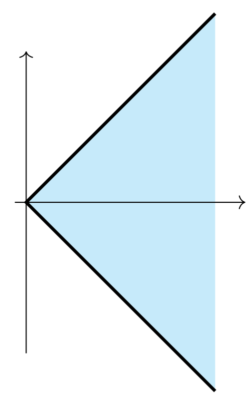
and

$$D = \{z\colon |\operatorname{Im}(z)| \le \operatorname{Re}(z)\}.$$

Find

$$\max_{z\in D} \operatorname{Im}(f(z)) \qquad \text{and} \qquad \min_{z\in D} \operatorname{Im}(f(z))$$

and point(s), where it is achieved.



Solution. Observe that $f(z) \to 0$ as $D \ni z \to \infty$ and therefore maximum and minimum are achieved at $z \in D$ (unless they are 0). By maximum principle both are achieved on the boundary.

Consider rays $y = \pm x,\ x > 0$, there

$$f(z) = \frac{1}{(1\pm i)^2x^2+1} = \frac{1}{\pm 2ix^2+1} = \frac{(1\mp 2ix^2)}{4x^4+1}$$
$$\implies \operatorname{Im}(f(z)) = \mp\frac{2x^2}{4x^4+1} =: \mp\varphi(x).$$

Then

$$\varphi'(x) = \frac{4x(1-4x^4)}{(4x^4+1)^2} = 0 \implies x = \frac{1}{\sqrt{2}}.$$

We need to consider only $x = \frac{1}{\sqrt{2}}$. Finally, $\varphi(x) = 0$ as $x = 0$. So,

$$\max_{z \in D} \operatorname{Im}(f(z)) = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \min_{z \in D} \operatorname{Im}(f(z)) = -\frac{1}{\sqrt{2}}$$

achieved at $z = \frac{1}{\sqrt{2}}(1 - i)$ and $z = \frac{1}{\sqrt{2}}(1 + i)$ respectively. □

3.2.2 Schwarz's Lemma

One of the best-known applications of the maximum-modulus principle is the following result, known for historical reasons as Schwarz's Lemma:

Theorem 3.2.7 (Schwarz's Lemma; Theorem 2 in the Textbook)**.** *Let $f(z)$ be analytic in the unit disk $D = \{z \colon |z| < 1\}$. Assume that $f(0) = 0$ and that $|f(z)| \le 1$ $\forall z \in D$. Then*

(i) The following inequality holds:

$$|f(z)| \le |z| \qquad \forall z \in D. \tag{3.2.2}$$

(ii) Equality can hold for some $z \neq 0$ only if $f(z) = \lambda z$ with a constant $\lambda \colon |\lambda| = 1$.

Proof. (i) Since $f(0) = 0$, we know that $g(z) = f(z)/z$ is also analytic in D. For $|z| = r$, we have $|g(z)| = \frac{|f(z)}{r} \le \frac{1}{r}$.

By the maximum-modulus principle, the inequality $|g(z)| \le 1/r$ is true for $|z| < r$ as well. Since r can be arbitrarily close to 1, we must conclude that $|g(z)| \le 1$ if $|z| < 1$; thus, $|f(z)| \le |z|$ in D.

(ii) Furthermore, if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $1 = |g(z_0)|$; consequently, $|g(z)|$ has an interior maximum, which is possible only if $g(z)$ is a constant λ with $|\lambda| = 1$. Then $f(z) = \lambda z$.

□

Remark 3.2.3. Schwarz's Lemma does not hold for functions of real variable. For example, $f(x) = \frac{2x}{x^2+1}$ satisfies $|u(x)| \le 1$ for all $x \in \mathbb{R}$ and u is smooth, $u(0) = 0$ but $u(x) > |x|$ for $x \in (-1, 1)$.

3.2.3 Mean Value Theorem

Theorem 3.2.8 (Mean-Value Theorem)**.** *Let $f(z)$ be analytic function in the disk $\{z\colon |z-z_0|<r\}$ and continuous in the closed disk $\{z\colon |z-z_0| <\leq r\}$. Then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt. \tag{3.2.3}$$

where $\gamma = \{z\colon |z-z_0| = r\}$.

Proof. Cauchy's Formula gives us

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z_0}$$

when γ is a circle and z_0 is inside γ. Plugging $\zeta = z_0 + re^{it}$ with $0 \leq t \leq 2\pi$ and $d\zeta = ire^{it}\,dt$, we prove (3.2.3). □

Taking a real part of (3.2.3) we arrive to

Theorem 3.2.9 (Mean-Value Theorem for harmonic functions)**.** *Let $u(x,y)$ be harmonic function in the disk $\{z\colon |z-z_0|<r\}$ and continuous in the closed disk $\{z\colon |z-z_0|\leq r\}$. Then*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,dt. \tag{3.2.4}$$

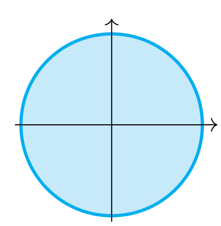
where $\gamma = \{z\colon |z-z_0| = r\}$.

Example 3.2.5.

Let u be a harmonic function in the disk $D = \{(x,y)\colon x^2+y^2 \leq 9\}$, equal to

$$g(\theta) = \frac{|\sin(\theta)|}{2+\cos(\theta)}$$

on its boundary, where θ is a polar angle. Find $u(0)$.



Solution.

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{|\sin(\theta)|\,d\theta}{2+\cos(\theta)} = \frac{1}{\pi}\int_0^{\pi} \frac{|\sin(\theta)|\,d\theta}{2+\cos(\theta)}$$
$$= -\frac{1}{\pi}\ln(2+\cos(\theta))\Big|_{\theta=0}^{\theta=\pi} = \frac{1}{\pi}\ln(3).$$

□

3.3 Linear Fractional Transformations

3.3.1 Introduction

In these couple of lectures we consider linear fractional transformations (a.k.a. Möbius transformations) which are one of the most basic examples of conformal transformations, which we consider after this. Here the introduction of the extended complex plane $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ will be especially fruitfull.

Definition 3.3.1. *Linear fractional transformation* is a map $T\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by

$$T(z) := \frac{az+b}{cz+d} \tag{3.3.1}$$

with $a, b, c, d \in \mathbb{C}$, satisfying $(ad - bc) \neq 0$.

Remark 3.3.1. Assumption $\Delta := ad - bc \neq 0$ is essential; otherwise $(a, b)]$ would be proportional to (b, c) and $T(z)$ would be a constant, equal to the coefficient of proportionality.

Theorem 3.3.1. *(i) Composition of two linear fractional transformations is a linear fractional transformation as well.*

(ii) Inverse to a linear fractional transformation is also a linear fractional transformation.

(iii) Identical transformation is also a linear fractional transformation.

Proof. (i) Let $w = T_1(z) = \dfrac{a_1 z + b_1}{c_1 z + d_1}$ and $T_2(w) = \dfrac{a_2 w + b_2}{c_2 w + d_2}$. Then

$$(T_2 \circ T_1)(z) = \frac{a_2 \dfrac{a_1 z + b_1}{c_1 z + d_1} + b_2}{c_2 \dfrac{a_1 z + b_1}{c_1 z + d_1} + d_2} = \frac{a_2(a_1 z + b_1) + b_2(c_1 z + d_1)}{c_2(a_1 z + b_1) + d_2(c_1 z + d_1)}$$
$$= \frac{(a_2 a_1 + b_2 c_1) z + (a_2 b_1 + b_2 d_1)}{(c_2 a_1 + d_2 c_1) + (c_2 b_1 + d_2 d_1)} = \frac{a_3 z + b_3}{c_3 z + d_3} \quad (3.3.2)$$

is a linear fractional transformation.

(ii) Let $w = T(z) = \dfrac{az + b}{cz + d}$. Then

$$w(cz + d) = (az + b) \implies (cw - a)z = (-wd + b)$$
$$\implies z = T^{-1}(w) = \frac{dw - b}{a - cw} \quad (3.3.3)$$

is a linear fractional transformation.

(iii) Finally, $T(z) = z$ is a linear fractional transformation with $a = 1, b = 0, c = 0, d = 1$.

□

Remark 3.3.2. Observe that to linear fractional transformation T with coefficients a, b, c, d one can assign a 2×2-matrix $M[T] \coloneqq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Then (3.3.2) shows that to $T_3 \coloneqq T_2 \circ T_1$ is assigned matrix

$$M[T_3] = \begin{pmatrix} a_2 a_1 + b_2 c_1 & a_2 b_1 + b_2 d_1 \\ c_2 a_1 + d_2 c_1 & c_2 b_1 + d_2 d_1 \end{pmatrix} = M[T_2] M[T_1].$$

(b) Further, (3.3.3) shows that to T^{-1} is assigned matrix

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \Delta M[T]^{-1} \qquad \text{with} \quad \Delta = ad - bc.$$

(c) Finally, to $T(z) = z$ is assigned matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(d) However, the same transform T corresponds to the whole bunch of quadruplets $\{a, b, c, d\}$, different by a common factor $\lambda \neq 0$. Requiring that

$$\Delta := ad - bc = 1 \tag{3.3.4}$$

would ensure that to each transform T correspond exactly two such quadruplets, different by factor -1, and thus two matrices $M[T]$ with determinant 1, different by a factor -1.

(e) Under this condition to T^{-1} is assigned matrix $M[T]^{-1}$.

(f) On the other hand, inverse map $M \mapsto T$ (where M is any 2×2-matrix with determinant 1) is single valued and according to Part **1**

(g) $T[M_2 M_1] = T[M_2] \circ T[M_1]$,

(h) $T[M^{-1}] = (T[M])^{-1}$,

(i) and $T[\pm I] = \mathsf{id}$ is an identical map $z \to z$.

We can say few mathematical jargon words showing how learned we are, but we are not in Math Specialist class.

Theorem 3.3.2. *A linear fractional transformation*

$$T(z) = \lambda \frac{a - z}{1 - \bar{a}z} \qquad \text{with } \; a, \lambda \in \mathbb{C}, \; |a| < 1 \text{ and } \; |\lambda| = 1 \tag{3.3.5}$$

maps a unit disk $\mathbb{D} := \{z \colon |z| < 1\}$ *onto itself.*

Proof. Observe that if T is defined by (3.3.5) then

$$|T(z)|^2 = \frac{|a - z|^2}{1 - \bar{a}z|^2} = \frac{|a|^2 - 2\,\mathrm{Re}(\bar{a}z) + |z|^2}{1 - 2\,\mathrm{Re}(\bar{a}z) + |a|^2|z|^2}$$

and $|T(z)| < 1$ if and only if

$$1 + |a|^2|z|^2 > |a|^2 + |z|^2 \iff (1 - |a|^2)(1 - |z|^2) > 0$$

which is true iff $|z| < 1$ that means $z \in \mathbb{D}$. Therefore, it maps $\mathbb{D}$ onto itself. $\square$

Theorem 3.3.3. *(i) Composition of two transformations of the type* (3.3.5) *is also a transformation of this type.*

(ii) Inverse of transformation (3.3.5) *is also a transformation of this type.*

(iii) Identity transformation is also a transformation of type (3.3.5).

Proof. (i) Let $w = T_1(z) = \lambda_1 \dfrac{z-a_1}{1-\bar{a}_1 z}$, $T_2(w) = \lambda_2 \dfrac{w-a_2}{1-\bar{a}_2 w}$; then

$$(T_2 \circ T_1)(z) = \lambda_2 \frac{\lambda_1 \frac{z-a_1}{1-\bar{a}_1 z} - a_2}{1 - \lambda_1 \frac{z-a_1}{1-\bar{a}_1 z}\bar{a}_2} = \lambda_2 \frac{\lambda_1(z-a_1) - a_2(1-\bar{a}_1 z)}{(1-\bar{a}_1 z) - \lambda_1(z-a_1)\bar{a}_2}$$

$$= \lambda_2 \frac{z(\lambda_1 + a_2\bar{a}_1) - (\lambda_1 a_1 + a_2)}{(1+\lambda_1 a_1 \bar{a}_2) - (\bar{a}_1 + \lambda_1 a_1 \bar{a}_2)z} = \lambda_3 \frac{z-a_3}{1-\bar{a}_3 z}$$

with $\lambda_3 = \lambda_2 \dfrac{\lambda_1 + a_2\bar{a}_1}{1+\lambda_1 a_1 \bar{a}_2}$, $a_3 = \dfrac{\lambda_1 a_1 + a_2}{1+\lambda_1 a_1 \bar{a}_2}$ (*Check calculations!*)

One can check easily that $|\lambda_3| = 1$ and $|a_3| < 1$:

$$|\lambda_3|^2 = |\lambda_2|^2 \frac{|\lambda_1 + a_2\bar{a}_1|^2}{|1+\lambda_1 a_1\bar{a}_2|^2} = \frac{|\lambda_1|^2 + 2\operatorname{Re}(\lambda_1\bar{a}_2 a_1) + |a_2|^2|a_1|^2}{1 + 2\operatorname{Re}(\lambda_1 a_1\bar{a}_2) + |\lambda_1|^2|a_1|^2|a_2|^2} = 1$$

because $|\lambda_1| = |\lambda_2| = 1$.

$$|a_3|^2 = \frac{|\lambda_1 a_1 + a_2|^2}{|1+\lambda_1 a_1\bar{a}_2|^2} = \frac{|\lambda_1|^2|a_1|^2 + 2\operatorname{Re}(\lambda_1 a_1\bar{a}_2) + |a_2|^2}{1 + 2\operatorname{Re}(\lambda_1 a_1\bar{a}_2|) + |\lambda_1|^2|a_1|^2|a_2|^2}$$

$$= \frac{|a_1|^2 + 2\operatorname{Re}(\lambda_1 a_1\bar{a}_2) + |a_2|^2}{1 + 2\operatorname{Re}(\lambda_1 a_1\bar{a}_2|) + |a_1|^2|a_2|^2} < 1$$

because $|a_1|^2 + |a_2|^2 < 1 + |a_1|^2|a_2|^2$ (since $|a_1| < 1, |a_2| < 1$).

(ii) If $w = \lambda \dfrac{z-a}{1-\bar{a}z}$ then $z = \lambda_2 \dfrac{w-a_2}{1-\bar{a}_2 w}$ with $\lambda_2 = \bar{\lambda}$ and $a_3 = -\lambda a$, $|\lambda_2| = 1$ and $|a_2| < 1$ (*Check it!*).

(iii) Finally $w = z$ is given by (3.3.5) with $\lambda = 1$, $a = 0$.

□

Theorem 3.3.4. *Let* $|z_1| < 1$, $|z_2| = 1$ *and* $|w_1| < 1$, $|w_2| = 1$. *Then exists exactly one transformation* L *of type* (3.3.5) *such that* $L(z_1) = a_1$ *and* $L(z_2) = w_2$.

Proof. Let us construct transformation T of the type (3.3.5) such that $T(z_2) = 0$ and $T(z_2) = 1$. It is uniquely given by

$$T(z) = \left(\frac{1 - \bar{z}_1 z_2}{z_2 - z_1}\right)\left(\frac{z - z_1}{1 - \bar{z}_1 z}\right). \tag{3.3.6}$$

Then uniquely $L = S^{-1} \circ T$ with $S(w)$ given by the same formula (3.3.5) with w, w_1 and w_2 instead of z, z_1 and z_2. Finally

$$\left(\frac{1 - \bar{z}_1 z_2}{z_2 - z_1}\right)\left(\frac{z - z_1}{1 - \bar{z}_1 z}\right) = \left(\frac{1 - \bar{w}_1 w_2}{w_2 - w_1}\right)\left(\frac{w - w_1}{1 - \bar{w}_1 w}\right). \tag{3.3.7}$$

□

Theorem 3.3.5. *Any linear fractional transformation T, mapping $\mathbb{D}$ onto $\mathbb{D}$ is transformation of the type* (3.3.5).

Proof. There are $z_1 \colon |z_1| < 1$ such that $T(z_1) = 0$ and $z_2 \colon |z_2| = 1$ such that $T(z_2) = 1$. Let S be a transformation of type (3.3.5) such that $S(z_1) = 0$ and $S(z_2) = 1$. Consider linear fractional transformation $L = T \circ S^{-1}$, it maps 0 onto 0, and therefore $L(z) = \dfrac{az + b}{cz + d}$ with $b = 0$; it can be rewritten as $L(z) = \dfrac{z}{c_1 z + d_1}$.

If it maps $\mathbb{D}$ onto $\mathbb{D}$ then $|z| = 1 \implies |c_1 z + d_1| = 1$ which is possible if either $|c_1| = 1$, $d_1 = 0$ or $c_1 = 0$, $|d_1| = 1$. In the former case $L(z) = c_1^{-1}$ (constant, which is impossible), in the latter $L(z) = d_1^{-1} z$.

Finally, $L(1) = 1 \implies d_1 = 1 \implies L = T \circ S^{-1} = \mathsf{id} \implies T = S$ of type (3.3.5). □

Remark 3.3.3. In the next Section 3.4, this proof will be given not just for linear fractional transformations, but for more general *conformal maps.*

3.3.2 Fixed Points and Triples to Triples

Definition 3.3.2. *A fixed point of the map $z \to T(z)$* is a point such that

$$T(z) = z. \tag{3.3.8}$$

Theorem 3.3.6. *A fractional linear transformation that is not identically equal to z has at most two distinct fixed points.*

Proof. z is a fixed point of fractional linear transformation $T(z) = \dfrac{az+b}{cz+d}$ with $c \neq 0$ if and only if z is a root of the quadratic equation

$$cz^2 + (d-a)z - b = 0$$

which has at most two distinct roots. Case $c = 0, d \neq 0$ consider by yourself. □

Corollary 3.3.7. *Let T and S be two fractional linear transformations that are equal at three distinct points: $T(z_j) = S(z_j)$ for $j = 1, 2, 3$. Then $T(z) = S(z)$ for all z.*

Proof. If $T(z_j) = S(z_j)$ then $(S^{-1} \circ T)(z_j) = z_j$ for $j = 1, 2, 3$ and since $S^{-1} \circ T$ is also a fractional linear transformation we conclude that $(S^{-1} \circ T)(z) = z$ for all z, and $T(z) = S(z)$ for all z. □

Theorem 3.3.8. *Let z_1, z_2 and z_3 be three distinct complex numbers, and let w_1, w_2 and w_3 be also three distinct complex numbers. Then there exists a unique linear fractional transformation L with $L(z_j) = w_j$ for $j = 1, 2, 3$.*

Proof. Let

$$T(z) = \left(\frac{z-z_1}{z-z_3}\right)\left(\frac{z_2-z_3}{z_2-z_1}\right). \tag{3.3.9}$$

Then $T(z_1) = 0$, $T(z_2) = 1$ and $T(z_3) = \infty$. Let us define $S(w)$ by the same formula (3.3.9) with w, w_1, w_2 and w_3 instead of z, z_1, z_2 and z_3. Then $S(w_1) = 0$, $S(w_2) = 1$ and $S(w_3) = \infty$. Then $L = S^{-1} \circ T$ is a required transformation. Due to Corollary 3.3.7 it is unique.

Finally,

$$\left(\frac{z-z_1}{z-z_3}\right)\left(\frac{z_2-z_3}{z_2-z_1}\right) = \left(\frac{w-w_1}{w-w_3}\right)\left(\frac{w_2-w_3}{w_2-w_1}\right). \tag{3.3.10}$$

□

Remark 3.3.4. We can extend these arguments to the case when one of z_1, z_2, z_3 or/and w_1, w_2, w_3 is ∞. Indeed, if $z_1 = \infty$, we can take $T(z) = \frac{z_2-z_3}{z-z_3}$; if $z_2 = \infty$, then $T(z) = \frac{z-z_1}{z-z_3}$; and if $z_3 = \infty$, then $T(z) = \frac{z-z_1}{z_2-z_1}$.

Example 3.3.1. Find a linear fractional transformation, that sends $0, 1, 2$ to $-1, 0, 4$ respectively.

Solution. Using (3.3.10) we have

$$\begin{aligned}&(\frac{z-0}{2-z})(\frac{1-2}{1-0}) = \frac{z}{z-2}\\ =&(\frac{w+1}{w-4})(\frac{0-4}{0+1}) = -4\frac{w+1}{w-4} \implies w = \frac{8z-8}{-3z+8}.\end{aligned}$$

□

Example 3.3.2. Find a linear fractional transformation, mapping $\mathbb{D}$ onto $\mathbb{D}$, that sends $\frac{1}{2}$, 1 to $\frac{1}{2}$, -1 respectively.

Solution. Using ((7))3.3.7) we have

$$\begin{aligned}&\left(\frac{1-\frac{1}{2}}{1-\frac{1}{2}}\right)\left(\frac{z-\frac{1}{2}}{1-\frac{z}{2}}\right) = \frac{2z-1}{z-2}\\ &\left(\frac{1+\frac{1}{2}}{-1-\frac{1}{2}}\right)\left(\frac{w-\frac{1}{2}}{1-\frac{w}{2}}\right) = -\frac{2w-1}{2-w} \implies w = \frac{5z-4}{4z-5}.\end{aligned}$$

□

3.3.3 Lines and Circles

Theorem 3.3.9. *A linear fractional transformation maps each circle onto another circle or onto a straight line and each straight line onto another straight line or a circle.*

Proof. First, if $T(z) = az+b$ with $a \neq 0$ (so T is a linear transformation), then T maps circles and straight lines to the circles and straight lines respectively: the circle $C = \{z\colon |z-z_0| = r\}$ is transformed to the circle $C' = \{w\colon |w-w_0| = |a|r\}$ with $w_0 = az_0+b$ and the straight line $L = \{z\colon \operatorname{Re}(Az+B) = 0\}$ is transformed to the straight line $L' = \{w\colon \operatorname{Re}\big((A/a)w + B - (Ab/a)\big) = 0\}$.

Thus, consider with $c \neq 0$

$$T(z) = \frac{az+b}{cz+d} = \frac{1}{c}\left[\frac{-\Delta}{cz+d} + a\right]$$

and we see that T is a composition of several simple transformations: $T = W \circ V \circ U$ with $U(z) = cz+d$, $V(w) = \dfrac{1}{w}$, and $W(\zeta) = \frac{1}{c}\big(-\Delta\zeta + a\big)$.

Since U and W are linear transformations, which map circles onto circles and straight lines onto straight lines, it is sufficient to prove the declared property for *inversion map* V. The equation

$$\alpha(x^2 + y^2) + \beta x + \gamma y = \delta$$

with real $\alpha, \beta, \gamma, \delta$ where $(\alpha, \beta, \gamma) \neq 0$ describes either a circle (if $\alpha \neq 0$ and $\beta^2 + \gamma^2 + 4\alpha\delta > 0$) or a straight line (if $\alpha = 0$). Since

$$\frac{1}{z} = \frac{x - yi}{x^2 + y^2} =: u + iv = w$$

dividing

$$\alpha(x^2 + y^2) + \beta x + \gamma y = \delta$$

by $x^2 + y^2$ we get

$$\alpha + \beta \frac{x}{x^2 + y^2} + \gamma \frac{y}{x^2 + y^2} = \frac{\delta}{x^2 + y^2}.$$

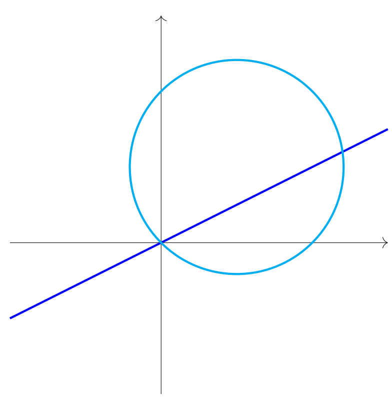
Plugging $u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$ we get

$$\alpha + \beta u - \gamma v = \delta(u^2 + v^2)$$

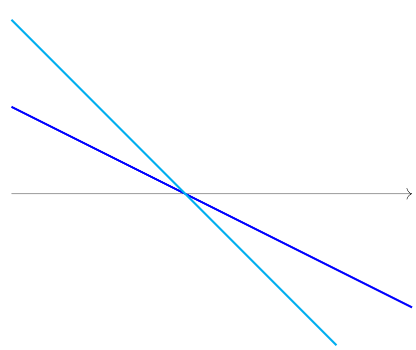
which also describes a circle (if $\delta \neq 0$) or a straight line (if $\delta = 0$). □

Remark 3.3.5. One can see easily that the inversion map transforms

(a) a straight line or a circle passing through the origin into a straight line,

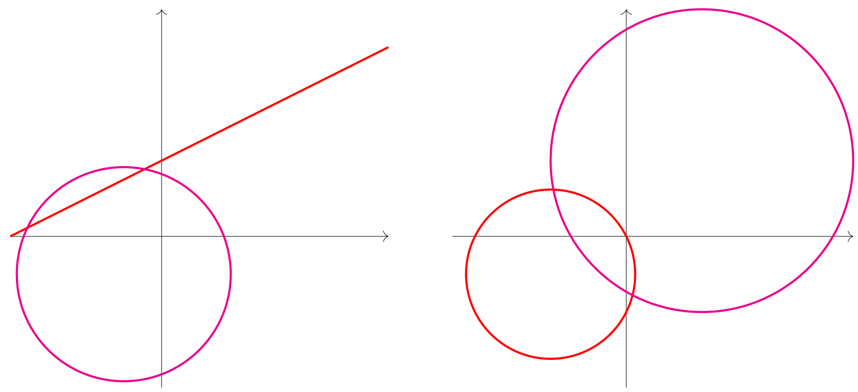


(a) z



(b) w

(b) a straight line or a circle not passing through the origin into a circle.



(a) z (b) w

3.4 Conformal Mapping

3.4.1 Introduction

Conformal maps are functions that preserve angles between curves (more precisely: between tangent lines to the curves in the points of intersection).

Let γ be the range of a smooth curve $z(t)$, passing through the point $z_0 = z(t_0)$, $a < t_0 < b$:

$$\gamma\colon z = z(t), \quad a < t < b, \qquad z(t_0) = z_0.$$

The curve γ has a tangent vector $z'(t_0)$ at z_0, and we suppose that $z'(t_0) \neq 0$.

Remark 3.4.1. If we consider another parametrization of the same curve γ, $z = z(t(s))$, $a' < s < b'$, then we will get another tangent vector, but it will be proportional with a positive coefficient if we assume that the direction in which curve is passed is the same for both parameterizations, that is, $t(s)$ is increasing function of s.

What happens to the curve γ when we apply *an analytic function* $f(z)$ to it? It is transformed into a new curve Γ in the w-plane; Γ is given by

$$\Gamma\colon w(t) = f(z(t)), \quad a < t < b,$$

Due to chain rule, the tangent vector to Γ at $w_0 = w(t_0) = f(z_0)$:

$$w'(t_0) = f'(z_0)z'(t_0). \tag{3.4.1}$$

In particular,

$$|w'(t_0)| = |f'(z_0)| \cdot |z'(t_0)|. \tag{3.4.2}$$

and

$$\arg(w'(t_0)) = \arg(f'(z_0)) + \arg(z'(t_0)). \tag{3.4.3}$$

Therefore under assumption

$$f'(z_0) \neq 0 \tag{3.4.4}$$

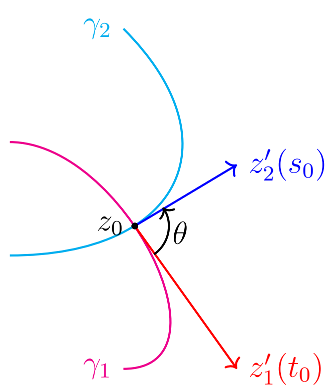
we conclude that

(a) *the tangent vector is scaled in length by a factor* $|f'(z_0)|$, and

(b) *the tangent vector is rotated by an angle* $\psi_0 = \arg(f'(z_0))$.

Definition 3.4.1. Let γ_1 and γ_2 be two smooth oriented curves that intersect at the point $z_0 = z_1(t_0) = z_2(s_0)$. Then *the angle between* γ_1 *and* γ_2 *at* z_0 *is the angle* θ *measured counter-clockwise* from the tangent vector $z_1'(t_0)$ to the tangent vector $z_2'(s_0)$, provided neither of these tangent vectors is zero:

$$\theta = \arg(z_2'(s_0)) - \arg(z_1'(t_0)). \tag{3.4.5}$$



Remark 3.4.2. This definition of the angle between two curves breaks down if one of the vectors $z_1'(t_0)$ or $z_2'(s_0)$ is 0.

Definition 3.4.2. Suppose now that $\phi(z)$ is a function, perhaps not analytic, defined in the disk $\{z\colon |z-z_0|<r$ and satisfying $\phi(z)\neq\phi(z_0)$ in the punctured disk $\{z\colon\ <|z-z_0|<r\}$.

Then map ϕ is *conformal at* z_0 if, whenever two oriented curves γ_1 and γ_2 meet at z_0 , the angle from $\Gamma_1=\phi(\gamma_1)$ to $\Gamma_2=\phi(\gamma_2)$ is equal to the angle from γ_1 to γ_2.

Remark 3.4.3. Mercator projection in geography also preserves angles (but stretches near polar domains more than near equatorial ones).

Theorem 3.4.1. *If f is analytic near z_0 and $f'(z_0)\neq 0$ then f is conformal at z_0.*

Proof. Indeed, if γ_1 and γ_2 intersect at z_0 then Γ_1 and Γ_2 intersect at $w_0=f(z_0)$ and equation (3.4.3)

$$\arg(w'(t_0)=\arg(f'(z_0))+\arg(z'(t_0)).$$

shows that the angle from Γ_1 to Γ_2 is equal to

$$\bigl[\arg(f'(z_0))+\arg(z_2'(s_0))\bigr]-\bigl[\arg(f'(z_0))+\arg(z_1'(t_0))\bigr]\\ =\arg(z_2'(s_0))-\arg(z_1'(t_0)),$$

which is an angle from γ_1 to γ_2. □

Remark 3.4.4. (a) If $f'(z_0)=0$ then f is not conformal. Indeed, assume that $f(z)-f(z_0)$ has 0 of multiplicity $m\geq 2$ at z_0. Then $f(z)-f(z_0)=g(z)^m$ with $g(z)$ also analytic, $g(z_0)=0$ and $g'(z_0)\neq 0$. Then g is conformal at z_0 but since $\arg(\zeta^m)=m\arg(\zeta)$ we conclude that at w_0 the angle from $\Gamma_1=f(\gamma_1)$ to $\Gamma_2=f(\gamma_2)$ equals to the angle from γ_1 to γ_2 at z_0, multiplied by m.

(b) However in the framework of **1** f is not one-to-one (it is m-to-1) which implies Theorem 3.4.2 below.

(c) Since the level lines of $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are orthogonal, we get another proof that level lines of $\mathrm{Re}(f(z))$ and $\mathrm{Im}(f(z))$ for any conformal f (that is analytic and $f'\neq 0$) are also orthogonal.

(d) Note that $z\mapsto f(\bar z)$ is *anti-conformal*: it preserves a magnitude of the angle but changes it sign.

(e) One can prove the converse to Theorem 3.4.1: if f is conformal, then it is analytic and $f' \neq 0$.

(f) One can wonder, if there is something like conformal mappings in higher dimensions. The answer is disappointing: there are but there are very few of them; those are shifts, rotations, scalings $\boldsymbol{x} \to \lambda\boldsymbol{x}$, mirror reflections, inversions $\boldsymbol{x} \to \frac{\boldsymbol{x}}{|\boldsymbol{x}|^2}$ and their compositions.

The more distant relatives are however plentiful.

Theorem 3.4.2. *If f is analytic on a domain D and one-to-one, then f is conformal at all points of domain D.*

The following assertion is important but even the rigorous statement is beyond our reach. You can find plenty counter-examples but they either are somehow pathological, or somehow could be fixed. So it is not a theorem, but more vague *principle*.

Extension of Conformal Map to the Boundary Principle. Let f be a conformal map of domain G_1 onto domain G_2. Then f maps the boundary Γ_1 of D_1 onto boundary Γ_2 of G_2.

3.4.2 Conformal Mappings of Unit Disk onto itself

Theorem 3.4.3. *All conformal maps of $\mathbb{D} = \{z\colon |z| < 1\}$ onto itself are of the form*

$$T(z) = \lambda\frac{z-a}{1-\bar{a}z} \qquad a, \lambda \in \mathbb{C}\colon |\lambda| = 1,\ |a| < 1. \tag{3.4.6}$$

Proof. We already know that (3.4.6) gives us a conformal map of $\mathbb{D}$ onto $\mathbb{D}$ and we need to prove that there are no other maps. Let $S\colon \mathbb{D} \to \mathbb{D}$ be some conformal map. Since it is one-to-one correspondence, it maps 0 to some point a. Consider map (3.4.6) with $\lambda = 1$. It maps $\mathbb{D} \to \mathbb{D}$ and a to 0. Therefore $L = T^{-1} \circ S$ also maps $\mathbb{D} \to \mathbb{D}$, and 0 to 0.

Since L maps its $\mathbb{D} \to \mathbb{D}$, it also maps boundary $\Gamma = \{z\colon |z| = 1\}$ onto itself: $|z| = 1 \implies |L(z)| = 1$. Therefore, due to Schwartz's Lemma $|L(z)| \le |z|$ for all $z\colon |z| < 1$ and, moreover, either

(a) $L(z) = \lambda z$ for some λ, or

(b) $|L(z)| < |z|$ for for all $z\colon 0 < |z| < 1$.

But case (b) is impossible. Indeed, we can apply the same arguments to inverse map L^{-1} and conculde that $|L^{-1}(w)| \le |w|$ for all $w\colon |w| < 1$ and plugging $w = L(z)$ we conclude that $|z| \le |L(z)| < |z|$ for all $z\colon 0 < |z| < 1$. Contradiction.

Therefore

$$(T^{-1} \circ S)(z) = \lambda z \implies S(z) = T(\lambda z) = \frac{\lambda z - a}{1 - \bar{a}\lambda z} = \lambda \frac{z - b}{1 - \bar{b} z}$$

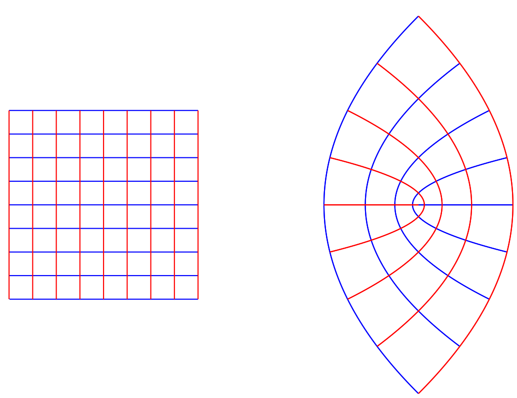
with $b = \bar{\lambda} a$ is indeed map of type (3.4.6). □

Corollary 3.4.4. *Let $z_1, w_1 \in \mathbb{D}$ and $z_2, w_2 \in \Gamma$. Then there exists exactly one conformal map T of $\mathbb{D}$ onto itself, such that $T(z_1) = w_1$ and $T(z_2) = w_2$.*

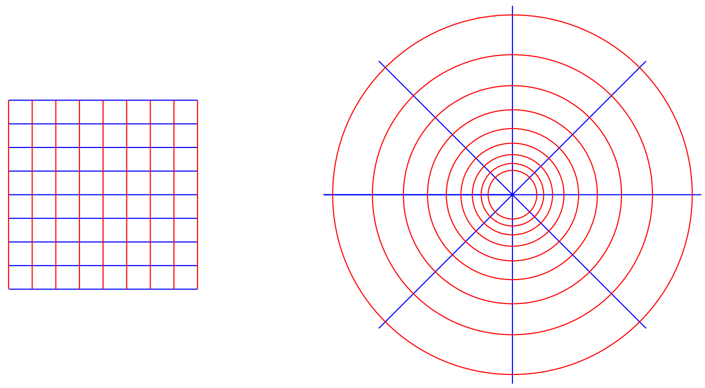
Proof. Indeed, any conformal map of $\mathbb{D}$ onto $\mathbb{D}$ is a map of type (3.4.6) and for them this statement has been proven. □

3.4.3 Examples

Example 3.4.1. $w = z^2$, it maps the right half-plane $\{z\colon -\frac{\pi}{2} < \arg(z) < \frac{\pi}{2}\}$ conformally onto plane with a cut $\{w\colon -\pi < \arg(w) < \pi\}$. What happens with level lines of z? $\{z = x + bi\}$ become parabolas $\{w = (x^2 - b^2) + 2bxi\}$ and $\{z = a + yi\}$ become also parabolas $\{w = (-y^2 + a^2) + 2ayi\}$:



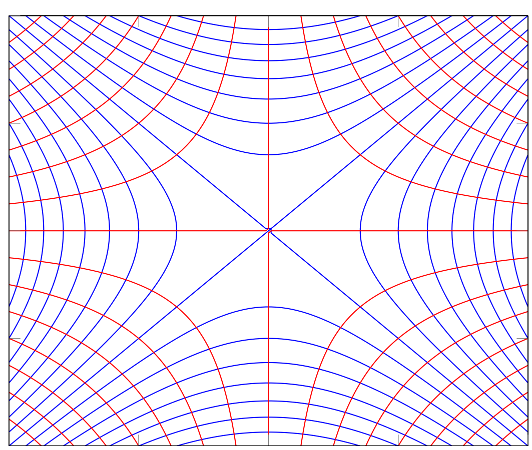
(a) z (b) w

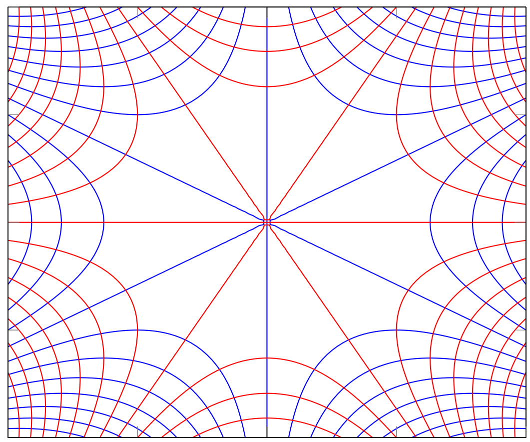


(a) z (b) w

Example 3.4.2. $w = e^z$; it maps a stripe $\{z\colon |\operatorname{Im}(z)| < \pi\}$ onto a plane with cut $\{w\colon -\pi < \arg(w) < \pi\}$. Since $z = x + yi$ maps onto $w = e^x(\cos(y) + i\sin(y))$ we conclude that $y = b$ become rays and $x = a$ became circles:

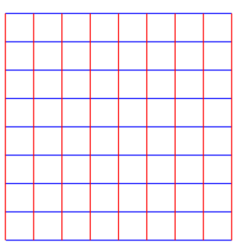
Example 3.4.3. $w = \sqrt{z}$, it maps the plane with a cut $\{z\colon -\pi < \arg(z) < \pi\}$ conformally onto right half-plane $\{w\colon -\frac{\pi}{2} < \arg(w) < \frac{\pi}{2}\}$, or a "double plane" (Riemannian surface, we leave exact definitions to math specialist class) $\{z\colon -2\pi < \arg(z) < 2\pi\}$ conformally onto a plane with a cut $\{w\colon -\pi < \arg(w) < \pi\}$. Here we look the other way: $w = u + iv$ and $z = u^2 - v^2 + 2uvi$, so we draw level lines of $u^2 - v^2$ and $2uv$, which are two families of hyperbolas.



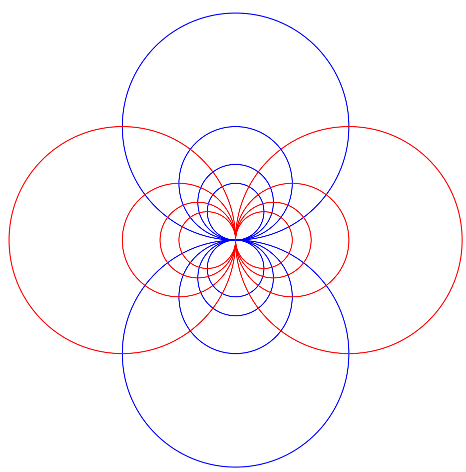


Example 3.4.4. $w = \sqrt[3]{z}$, it maps the plane with a cut $\{z\colon -\pi < \arg(z) < \pi\}$ conformally onto sector $\{w\colon -\frac{\pi}{3} < \arg(w) < \frac{\pi}{3}\}$, or a "tripple plane" (Riemannian surface, we leave exact definitions to math specialist class) $\{z\colon -3\pi < \arg(z) < 3\pi\}$ conformally onto a plane with a cut $\{w\colon -\pi < \arg(w) < \pi\}$. Here we look at level lines: $w = u + iv$ and $z = (u^3 - 3uv^2) + (3u^2v - v^3)i$, so we draw level lines of $u^3 - 3uv^2$ and $(3u^2v - v^3)i$.

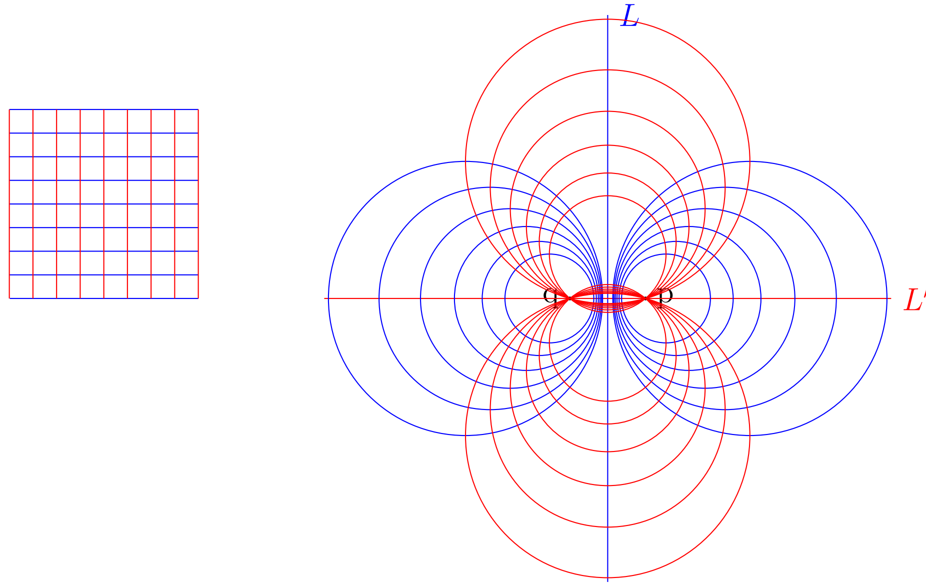
Example 3.4.5. $w = \frac{1}{z}$, it maps the punctured plane $\{z\colon |z| > 0\}$ conformally onto the punctured plane $\{w\colon |w| > 0\}$; we know that the straight lines are mapped onto circles, in particular, lines $\{z = x + bi\}$ onto circles $\{|w + (2b)^{-1}i| < |2b|^{-1}\}$ and $\{z = a + yi\}$ onto circles $\{|w - (2a)^{-1}| < |2a|^{-1}\}$.



(a) z



(b) w



(a) z (b) w

Example 3.4.6. $z = \log\left(\frac{w-p}{w-q}\right)$ with $p \neq q$, it is defined on the twice punctured plane $\{w\colon |w-q| > 0,\ |w-q| > 0\}$. Consider inverse map; then we get Circles of Appolonius. Indeed $\operatorname{Re}(z) = a$ means that $\frac{|w-p|}{|w-q|} = e^a$ and we get a family of blue circles, while $\operatorname{Im}(z) = b$ means that $\arg(z-p) - \arg(z-q) = b$ and we get a family of arcs of red circles between p and q (see W2L1).

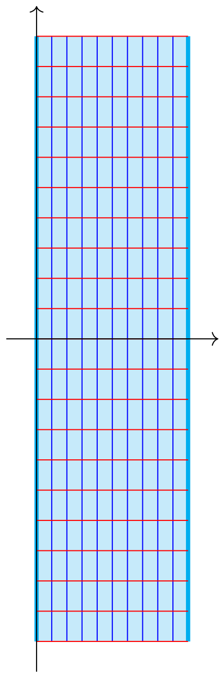
Example 3.4.7. $w = \cos(z)$. It maps conformally stripe $\{z\colon 0 < \operatorname{Re}(z) < \pi\}$ onto plane with cuts $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ (see W3L2):

(a) Vertical lines $\{z\colon x = \operatorname{Re}(z) = \mathsf{const},\ y = \operatorname{Im}(z)\}$ are mapped to $w = u + iv$ with $v = -\sin(x)\sinh(y)$, $u = \cos(x)\cosh(y)$ which are branches of hyperbolas satisfying

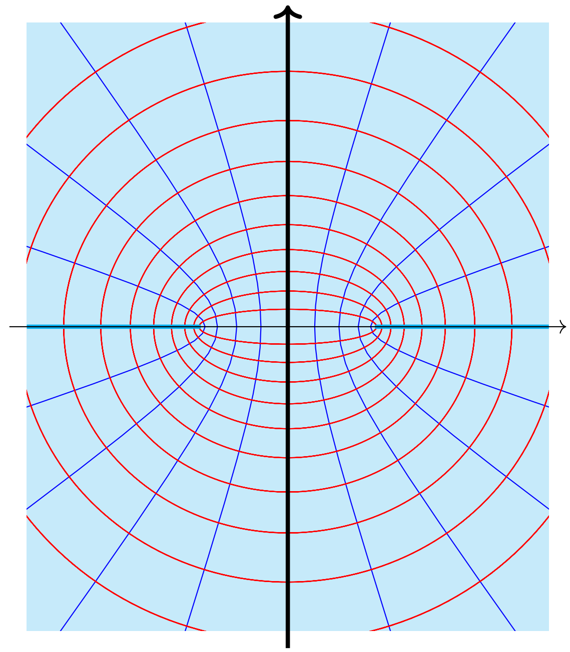
$$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1,$$

with $A = |\cos(x)|$, $B = \sin(x)$.

(b) Horizontal segments $\{z\colon 0 < x = \operatorname{Re}(z) < \pi,\ y = \operatorname{Im}(z) = \mathsf{const}\}$ are mapped to $w = u + iv$ with $v = -\sin(x)\sinh(y)$, $u = \cos(x)\cosh(y)$ which are halves of ellipses satisfying



(a) z



(b) $w = \cos(z)$

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$$

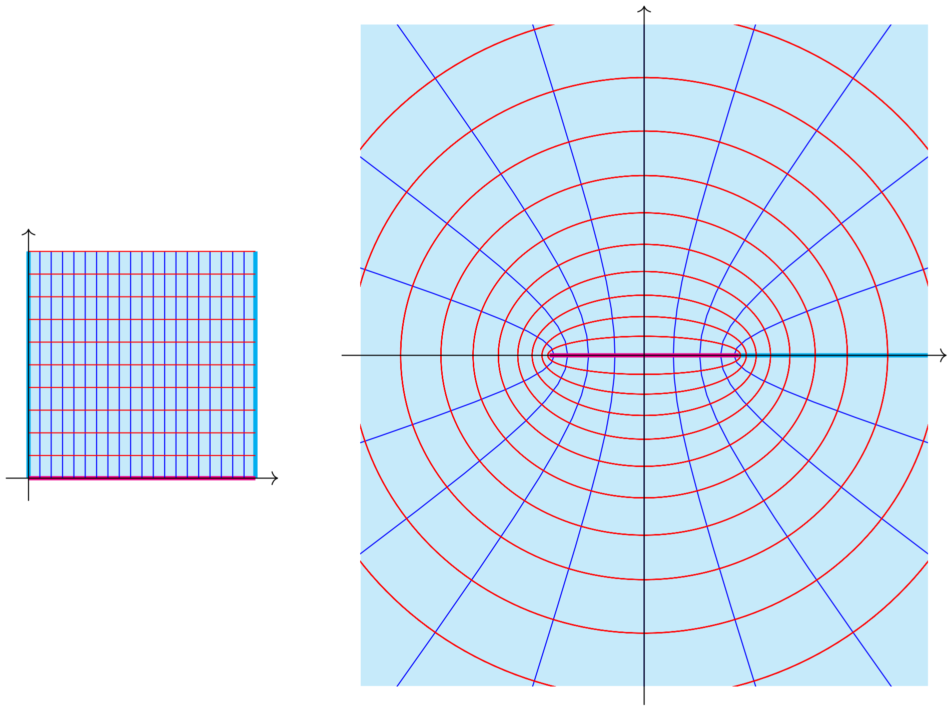
with $a = \cosh(y)$, $b = |\sinh(y)|$.

Example 3.4.8. $w = \cos(z)$. It maps conformally half-stripe $\{z\colon 0 < \mathrm{Re}(z) < 2\pi,\ \mathrm{Im}(z) > 0\}$ onto plane with a cut $\mathbb{C} \setminus [-1, \infty)$.

(a) Vertical half-lines $\{z\colon x = \mathrm{Re}(z) = \mathsf{const},\ y = \mathrm{Im}(z)\}$ are mapped to $w = u + iv$ with $v = -\sin(x)\sinh(y)$, $u = \cos(x)\cosh(y)$ which are half-branches of the same hyperbolas as in Example 3.4.7.

(b) Horizontal segments $\{z\colon 0 < x = \mathrm{Re}(z) < 2\pi,\ y = \mathrm{Im}(z) = \mathsf{const}\}$ are mapped to $w = u + iv$ with $v = -\sin(x)\sinh(y)$, $u = \cos(x)\cosh(y)$ which are the same ellipses as in Example 3.4.7.

Example 3.4.9 (Zhukovsky transform). $w = \frac{1}{2}\bigl(z + \frac{1}{z}\bigr)$. It maps $\mathbb{D}$ onto $\mathbb{C} \setminus [-1, 1]$. Indeed, circles $z = re^{it}$ with fixed r and $t \in [0, 2pi)$are mapped onto confocal ellipses $u = A\cos(t)$, $v = -B\sin(t)$ with $A = \frac{1}{2}(r + r^{-1})$,



(a) z (b) $w = \cos(z)$

$B = \frac{1}{2}(r^{-1} - r)$ and segments $z = e^{-s}e^{it}$ with fixed t and $s \in (-\infty, 0]$ are mapped onto confocal hyperbolae $u = a\cosh(s)$, $v = -b\sinh(s)$ with $a = \cos(t)$, $b = \sin(t)$.

3.5 The Riemann Mapping Theorem and Schwarz-Christoffel Transformations

3.5.1 The Riemann Mapping Theorem

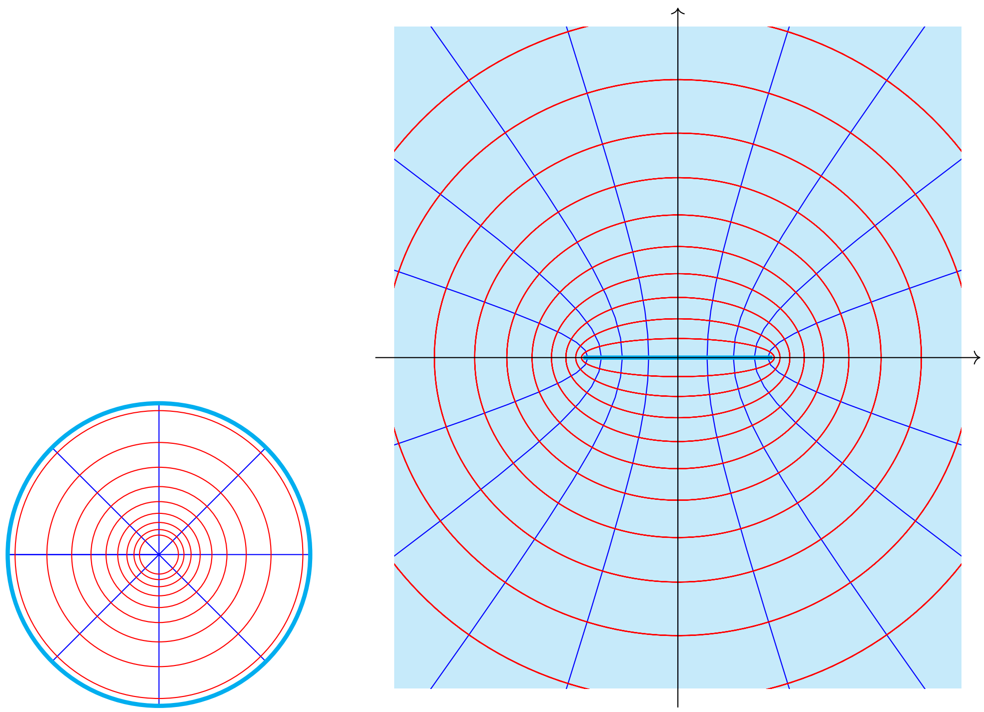
The following theorem is one of the most important theorems in the theory of conformal mappings:

Theorem 3.5.1 (The Riemann Mapping Theorem)**.** *Let G be a simply-connected domain in the extended complex plane $\widehat{\mathbb{C}}$ with the boundary, containing at least 2 disjoint points. Then there exists a conformal map from G onto $\mathbb{D}$.*

Remark 3.5.1. (a) The proof of this theorem is completely out of our reach. Actually there are many proofs and none is constructive.

(b) If the boundary of simply-connected domain contains at least two points, it contains continuum of them.

(c) We discuss which domains fit conditions of The Riemann Mapping Theorem.



(a) z (b) $w = (z + z^{-1})/2$

While The Riemann Mapping Theorem says how *domains are mapped*, it does not discuss the mapping of their boundaries. This is covered by Carathéodory's theorem below. But first we need the following

Definition 3.5.1. *Jordan curve* is a simple (without self-intersections) closed continuous curve (that means that there exists a one-to-one continuous map from the unit circle $\{z \colon |z| = 1\}$ onto Jordan curve.

Jordan curve has an important property, which looks obvious but in fact is very non-trivial and deep:

Theorem 3.5.3 (Jordan Curve Theorem)**.** *Let γ be a Jordan curve in the plane $\mathbb{R}^2$. Then its complement, $\mathbb{R}^2 \setminus \gamma$, consists of exactly two connected components. One of these components is bounded (the interior) and the other is unbounded (the exterior), and the curve γ is the boundary of each component.*

Again, we cannot prove it in this class.

Theorem 3.5.4 (Carathéodory's theorem)**.** *If f maps the open unit disk $\mathbb{D}$ conformally onto a bounded domain G in $\mathbb{C}$, then f has a continuous one-to-one extension to the closed unit disk if and only if its boundary $\gamma = \partial G$ is a Jordan curve.*

And again, we cannot prove it in this class.

3.5.2 Mirror Continuation

Theorem 3.5.5. *Let $f\colon D_1 \to D_2$ be a conformal map. Assume that their boundaries Γ_1 and γ_2 respectively, contain straight segments or circular arcs γ_1 and γ_2 correspondingly, and $f\colon \gamma_1 \to \gamma_2$.*

Then f could be extended throughout γ_1 by formula

$$\tilde{f}(z) = M_{\gamma_2} \circ f \circ M_{\gamma_1} \tag{3.5.1}$$

where M_γ is defined on the next frame.

Proof. It follows from:

(a) $f\colon \gamma_1 \to \gamma_2$;

(b) M_γ is *anticonformal map*, that is it preserves magnitude of angles, but changes orientation.

□

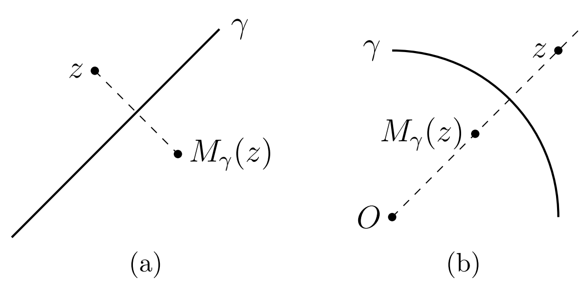


Figure 3.1: On (a) z and $M_\gamma(z)$ are on the equal distances from γ; if $\gamma \subset \mathbb{R}$ then $M_\gamma(z) = \bar{z}$. On (b) O is a center of the circle, R its radius and $|OM_\gamma(z)| = R^2|Oz|^{-1}$; if $\gamma \subset \{z\colon |z| = R\}$ then $M_\gamma(z) = \dfrac{R^2}{\bar{z}}$.

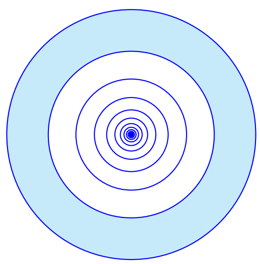
Corollary 3.5.6. *Let $D_1 = \{z\colon r_1 < |z| < R_1\}$ and $D_2 = \{w\colon r_2 < |w| < R_2\}$ and $f\colon D_1 \to D_2$ conformal one-to-one map of D_1 onto D_2. Then*

(i) $\dfrac{r_1}{R_1} = \dfrac{r_2}{R_2}$;

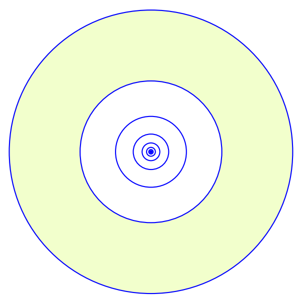
(ii) Either $f(z) = \lambda|z|$ *or* $f(z) = \dfrac{\mu}{z}$ *with* $|\lambda| = \dfrac{r_2}{r_1} = \dfrac{R_2}{R_1}$ *and* $|\mu| = r_1R_2 = r_2R_1$.

Proof. Assume that f maps inner circle onto inner circle and outer circle onto outer circle. Otherwise we replace $f(z)$ by $f(z)^{-1}$, r_2 by R_2^{-1} and R_2 by r_2^{-1}.

Using mirror reflection we conclude that f maps conformally punctured disk $\{z\colon 0 < |z| < R_1\}$ onto punctured disk $\{w\colon 0 < |w| < R_2\}$ and therefore disk onto disk, and $f(0) = 0$. Then $f(z) = \lambda z$ and $|\lambda| = R_1^{-1}R_2$:



(a) z



(b) w

Since f maps inner circle onto inner circle, it should be also $|\lambda| = r_1^{-1}r_2$, and it implies both (i) and (ii). □

3.5.3 Examples

Example 3.5.1 (Upper half-plane to unit circle). (a) Consider the following conformal map from $\mathbb{H}_+ := \{z\colon \operatorname{Im}(z) > 0\}$ onto $\mathbb{D} = \{w\colon |w| < 1\}$. According to Carathéodory's theorem it's boundary, $\{z\colon \operatorname{Im}(z) = 0\}$ should be mapped onto $\{w\colon |w| = 1\}$ and one of the way to achieve it is to take

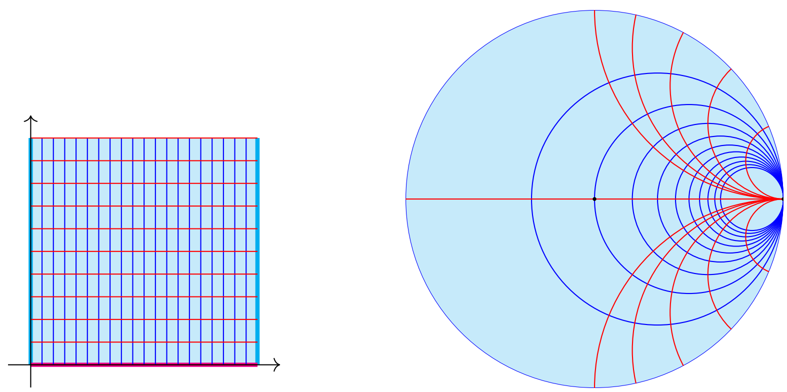
$$w = f(z) = \frac{z-i}{z+i}$$

we took "−" in the numerator and "+" in the denominator because otherwise it would be singular at $i \in \mathbb{H}_+$. And we can see that if $z = x + yi$ with $y > 0$, then $|w| < 1$ (check it!).

Further, since it is a linear fractional transformation, it must transform lines $\{z\colon \operatorname{Im}(z) = b\}$ into either circles or straight lines, but those must be circles, passing through 1 (when $|z| \to \infty$, $w \to 1$).

And indeed:

$$w = f(z) = \frac{x + (y-1)i}{x + (y+1)i} = \frac{[x + (y-1)i][x - (y+1)i]}{x^2 + (y+1)^2}$$



(a) z (b) w

(b) Also, the general conformal map $\mathbb{H}_+ \to \mathbb{D}$ is given by

$$w = \lambda \frac{z-a}{z-\bar{a}}, \qquad |\lambda| = 1,\ \operatorname{Im}(a) > 0.$$

(c) Inverse map then

$$z = \frac{\bar{a}w - \lambda a}{w - \lambda}, \qquad |\lambda| = 1,\ \operatorname{Im}(a) > 0.$$

Example 3.5.2. General conformal map $\mathbb{H}_+ \to \mathbb{H}_+$ is

$$w = a + \frac{c}{z-b}, \qquad a, b, c \in \mathbb{R},\ c > 0.$$

It does not include linear map $w = cz + a$ with $a, c \in \mathbb{R},\ c > 0$ which also maps $\mathbb{H}_+$ onto $\mathbb{H}_+$.

Example 3.5.3. $w = z^\beta$ maps sector $\{z \colon |\arg(z)| < \alpha\}$, $0 \neq \beta \in \mathbb{R}$ onto sector $\{w \colon |\arg(w)| < \beta\alpha\}$, $\alpha \leq \pi$, $|\beta|\alpha \leq \pi$. If $\beta > 0$ it maps 0 onto 0.

The general map from the first sector onto the second one does not preserve 0. F.e. $w = f(z) := z^2$ maps $Q = \{z \colon \operatorname{Im}(z) > 0,\ \operatorname{Re}(z) > 0\}$ onto $\mathbb{H}_+$, and the general conformal map $Q \to Q$ is $f^{-1} \circ T \circ f$ with T from Example 3.5.2.

Other examples were given before, see also pp 226–227 of the Textbook

3.5.4 Schwarz-Christoffel Transformations

A Schwarz-Christoffel transformation is an analytic conformal mapping of the upper half-plane onto a polygon.

The key to understanding it is the examination of the behaviour at the point x_0 of the function f given by

$$f(z) = A(z - x_0)^\beta + B$$

where x_0 and β are real numbers, $0 < \beta < 2$, and $0 \neq A$ and B are complex numbers.

Basically it is a composition of the linear function $w = A\omega + B$ which shifts and rotates, $\omega = \zeta^\beta$ which maps upper half-plane onto sector $\{\omega\colon 0 < \arg(\omega) < \alpha\pi\}$ and a shift along real axis $\zeta = z - x_0$.

Observe that $f'(z) = \beta A(z - x_0)^\alpha$ with $\alpha = \beta - 1 \implies -1 < \alpha < 1$.

Consider

$$f'(z) = A(z - x_1)^{\alpha_1} \cdots (z - x_N)^{\alpha_N} \tag{3.5.2}$$

with $x_1 < x_2 < \ldots < x_N$ and $|\alpha_j| < 1$ (so $f(z)$ will be primitive of it). Then

$$x > x_N \implies \arg(f'(x)) = \arg(A), \tag{3.5.3}$$
$$x_N > x > x_{N-1} \implies \arg(f'(x)) = \arg(A) + \pi\alpha_N, \tag{3.5.4}$$
$$x_{j+1} > x > x_j \implies \arg(f'(x)) = \arg(A) + \pi\alpha_{j+1} + \ldots + \pi\alpha_N, \tag{3.5.5}$$
$$x < x_1 \implies \arg(f'(x)) = \arg(A) + \pi\alpha_1 + \ldots + \pi\alpha_N. \tag{3.5.6}$$

Now we apply this to a mapping $\mathbb{H}_+$ onto polygon. Assume that it has vertices $w_0, w_1, \ldots, w_n$ listed counter-clockwise. And let $\theta_0, \theta_1, \ldots, \theta_N$ be *exterior angles*:

Note that

$$\theta_0 + \theta_1 + \ldots + \theta_N = 2\pi \tag{3.5.7}$$

and for $\alpha_j = -\theta_j/\pi$

$$-1 < \alpha_j < 1, \qquad \alpha_0 + \alpha_1 + \ldots + \alpha_N = -2. \tag{3.5.8}$$

Theorem 3.5.7 (Schwarz-Christoffel Theorem)**.** *Let P be a polygon in the plane with vertices $w_0 \ldots w_N$ and corresponding exterior angles θ_0, θ_1, ..., θ_N.*

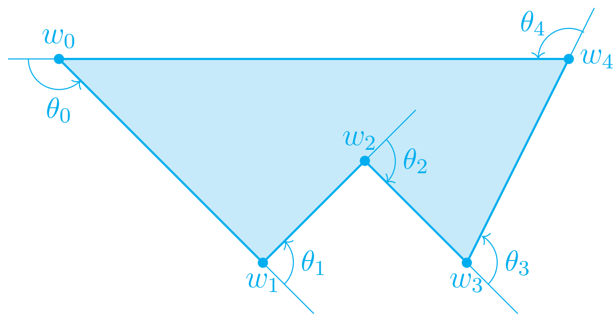


Figure 3.2: Note directions of angles!

Let $\alpha_j = -\theta_j/\pi$. *Then there exist real numbers* $x_1 < x_2 < \ldots < x_N$ *and a constant* A *such that the function* $f(z)$ *whose derivative is*

$$f'(z) = A(z - x_1)^{\alpha_1} \cdots (z - x_N)^{\alpha_N}, \tag{3.5.2}$$

which gives a one-to-one analytic mapping of the upper half-plane $\mathbb{H}_+ = \{z\colon \operatorname{Im}(z) > 0\}$ *onto the polygon* P*;* f *maps* x_j *to* w_j *for* $j = 1, \ldots, N$ *and*

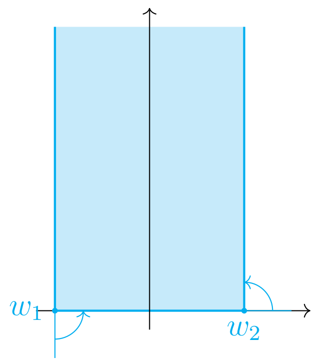
$$f(\infty) = \lim_{x\to\pm\infty} f(x) = w_0. \tag{3.5.9}$$

Remark 3.5.2. (a) The problem however is to find a primitive of (3.5.2) and then $x_1, \ldots, x_N$ and A.

(b) Polygon in this content is something more general, than the bounded polygon, leave alone bounded convex polygon, considered in geometry. It can be unbounded and contain cuts.

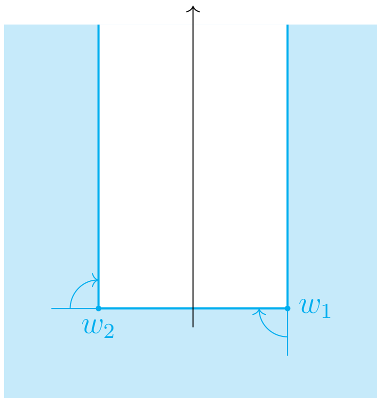
(c) While there are plenty of examples in the Textbook, we consider few, those where we can achieve a simple answer, may be even we already know.

Example 3.5.4. Let $P = \{w\colon -a < \operatorname{Re}(w) < a,\ \operatorname{Im}(w) > 0\}$.



Note that there is no w_0 (or rather $w_0 = \infty$ but we ignore it), $\theta_1 = \theta_2 = \frac{\pi}{2}$ and $\alpha_1 = \alpha_2 = -\frac{1}{2}$.
In contrast to (3.5.8) $\alpha_1 + \alpha_2 = -1$.
Let us take $x_{1,2} = \mp 1$. We can always rescale and shift what we got. Then $f'(z) = \dfrac{B}{\sqrt{1-z^2}}$ and $w = f(z) = B \arcsin(z) + C$.
And indeed, $z = \sin(w)$ maps strip $\{w\colon -\frac{\pi}{2} < \operatorname{Re}(w) < \frac{\pi}{2}, \operatorname{Im}(w) > 0\}$ onto $\mathbb{H}_+$ (so $B = 1$, $C = 0$).

Example 3.5.5. Let $P = \mathbb{C} \setminus \{w\colon -a < \operatorname{Re}(w) < a,\ \operatorname{Im}(w) > 0\}$.

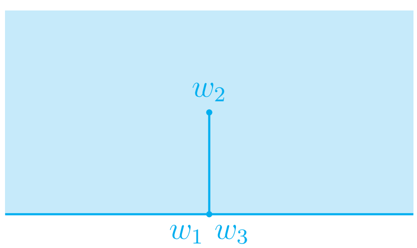


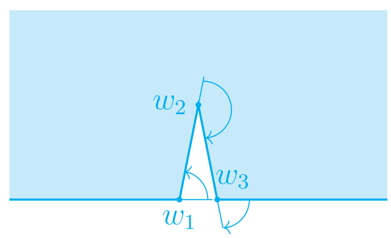
Note that there is no w_0 (or rather $w_0 = \infty$ but we ignore it), and w_1, w_2 are counter-clockwise (remember, how we go around domain), and now $\theta_1 = \theta_2 = -\frac{\pi}{2}$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$.
Again, in contrast to (3.5.8) $\alpha_1 + \alpha_2 = 1$. Let us take $x_{1,2} = \mp 1$. We can always rescale and shift what we got.
Then $f'(z) = 2B\sqrt{1 - z^2}$ and

$$w = f(z) = B\bigl(z\sqrt{1 - z^2} + \arcsin(z)\bigr) + C.$$

It is a nice exercise to prove that with $B = -1, C = 0$ maps $\mathbb{H}_+$ onto P with $a = \frac{\pi}{2}$.

Example 3.5.6. $P = \mathbb{H}_+ \setminus [0, i]$.





Here w_1 and w_3 are different points and $\theta_1 = \theta_3 = \frac{\pi}{2}$, $\theta_2 = -\pi$. To understand why so, look on the picture on the right. Then $\alpha_1 = \alpha_3 = -\frac{1}{2}$, $\alpha_2 = 1$ and selecting $x_{1,3} = \mp 1$, $x_2 = 0$ (symmetry suggest that it is a good idea) we get $f'(z) = \dfrac{Az}{\sqrt{z^2 - 1}} \implies f(z) = A\sqrt{z^2 - 1}$.

And indeed, with $A = 1$ we get a required map.