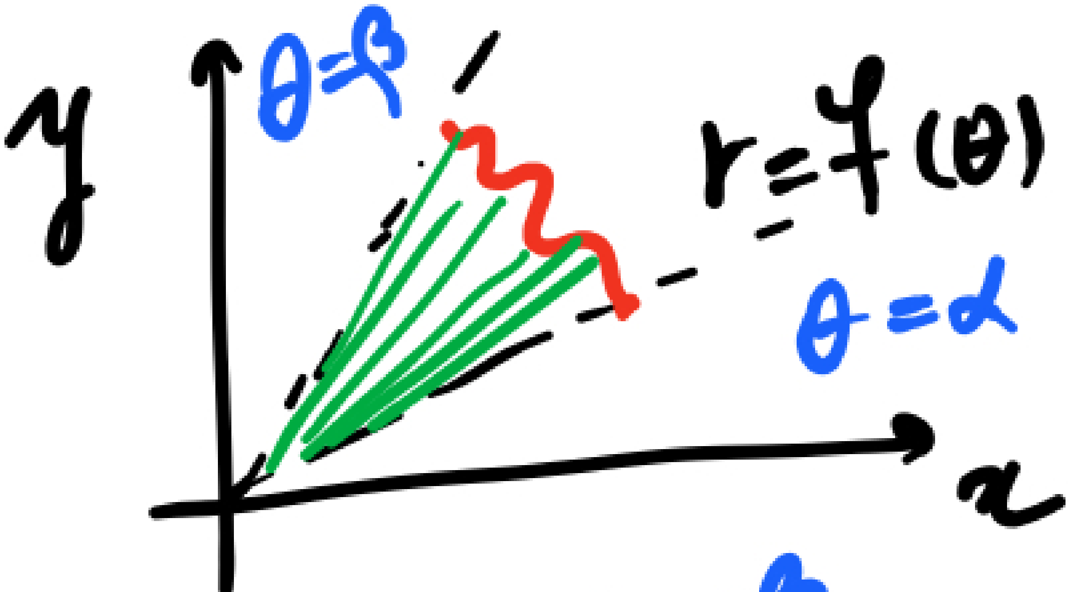
10.4. Areas and Lengths in Polar Coordinates.

- Areas swept out by polar curves



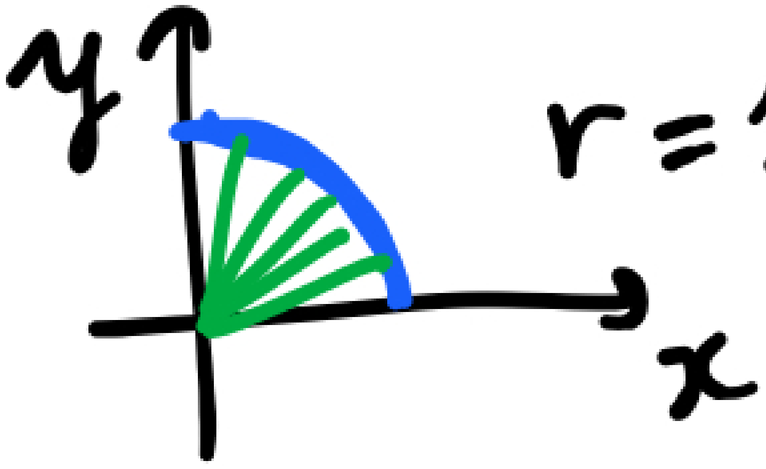
$\Delta A \approx \frac{1}{2} r^2 \Delta\theta$

$\alpha \le \theta \le \beta$

$$\boxed{A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta}$$

Examples

(1)



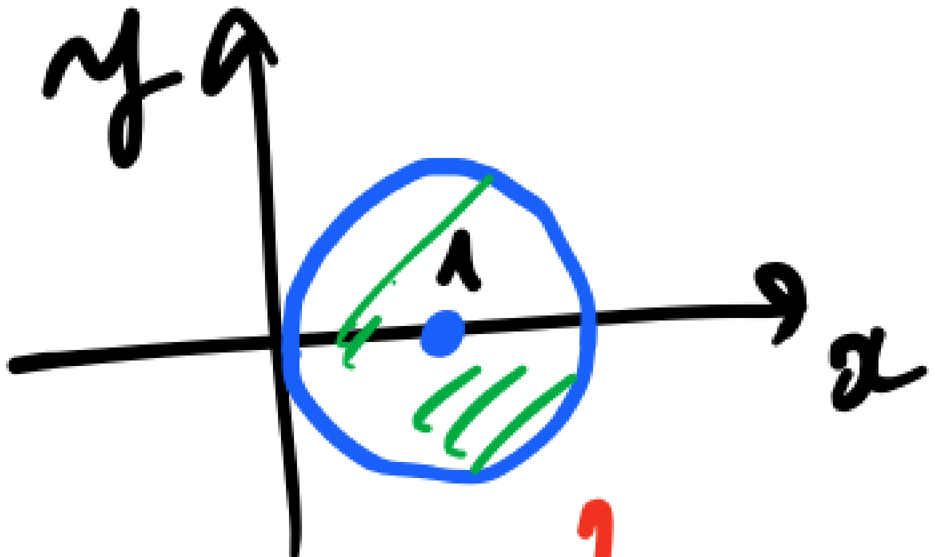
$r = 1$

$0 \le \theta \le \pi/2$

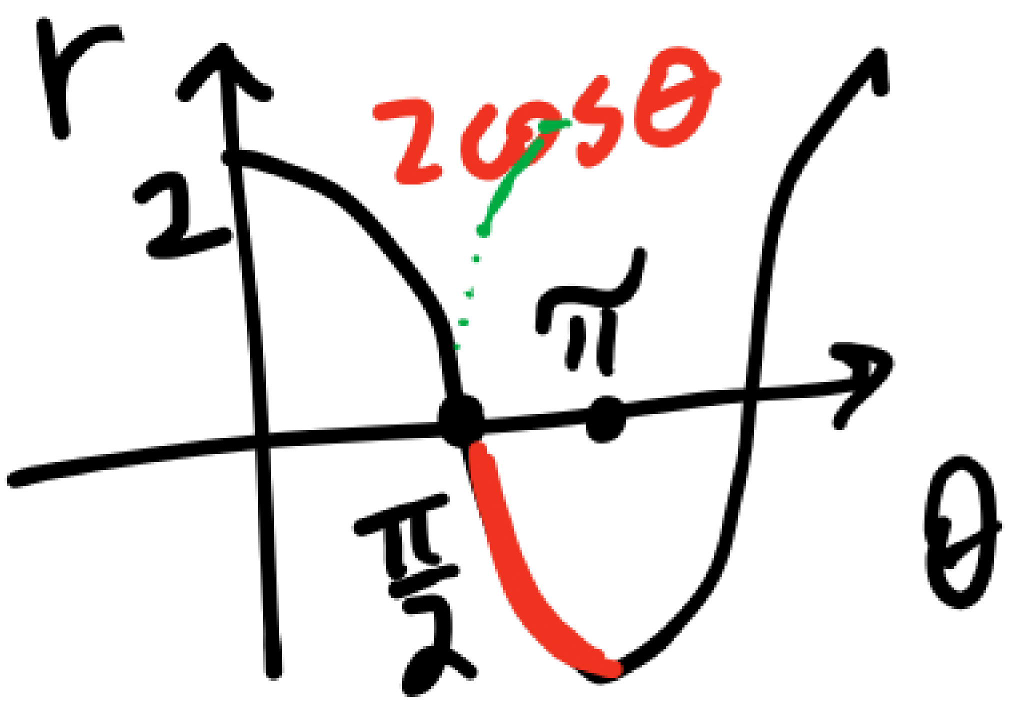
$$A = \int_0^{\pi/2} \frac{1}{2} 1^2 \, d\theta = \frac{\theta}{2} \Big|_0^{\pi/2} = \boxed{\frac{\pi}{4}}$$

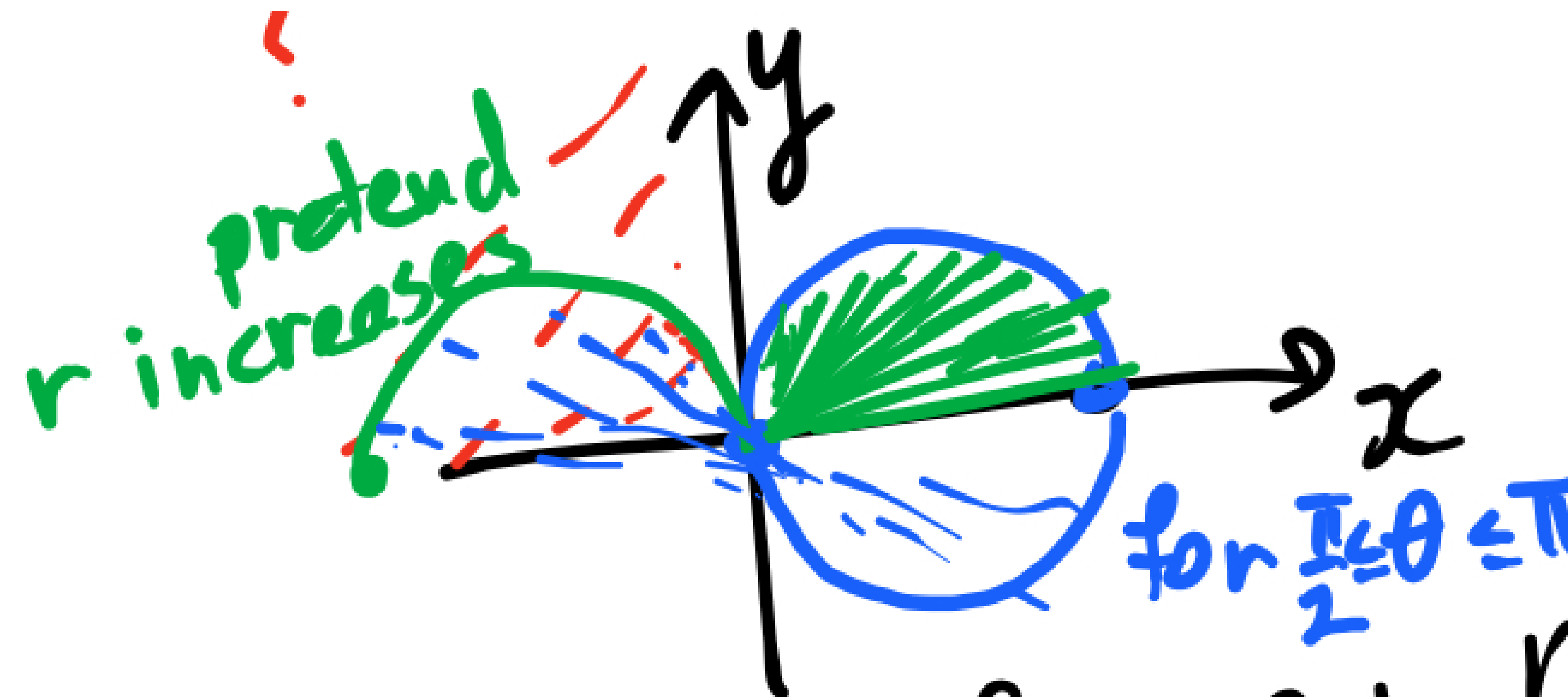
(2)

$r = 2\cos\theta$



$$A = \int \frac{1}{2} r^2 \, d\theta = \int_{?}^{?} \frac{1}{2} (2\cos\theta)^2 \, d\theta$$





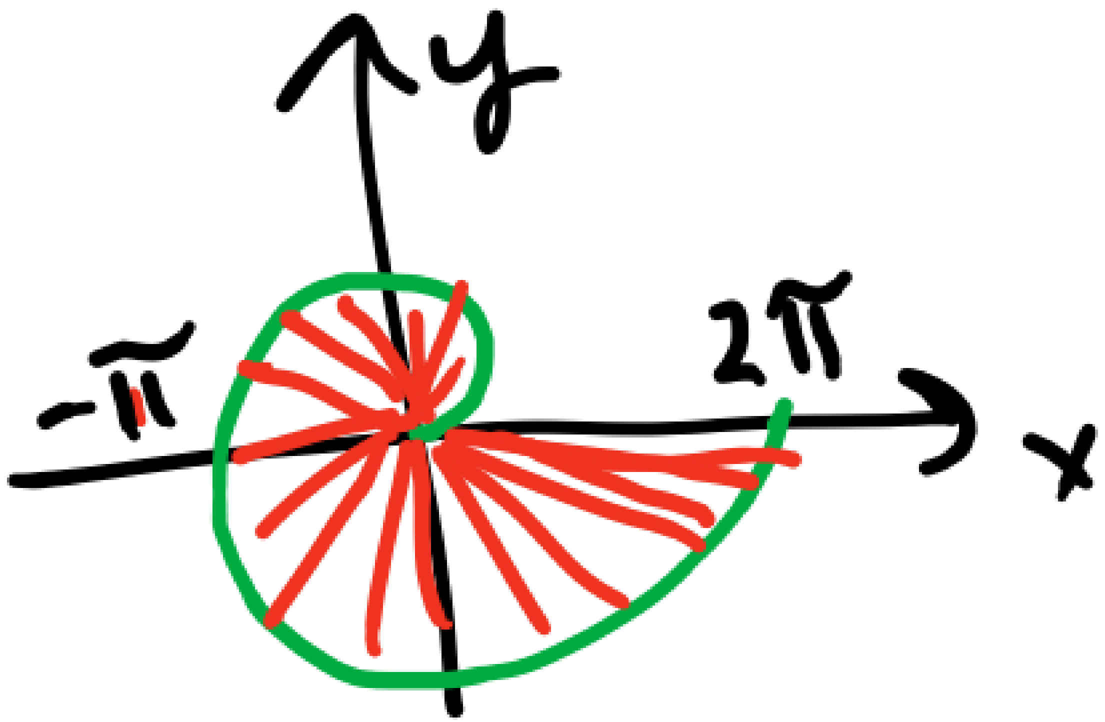
for $0 \le \theta \le \frac{\pi}{2}$, r decreases from 2 to 0

for $\frac{\pi}{2} \le \theta \le \pi$, r decreases from 0 to -2

check!

$$A = \int_0^{\pi} \frac{1}{2}(2\cos\theta)^2\,d\theta = \dots = \pi$$

(3) $r = \theta$



$0 \le \theta \le 2\pi$

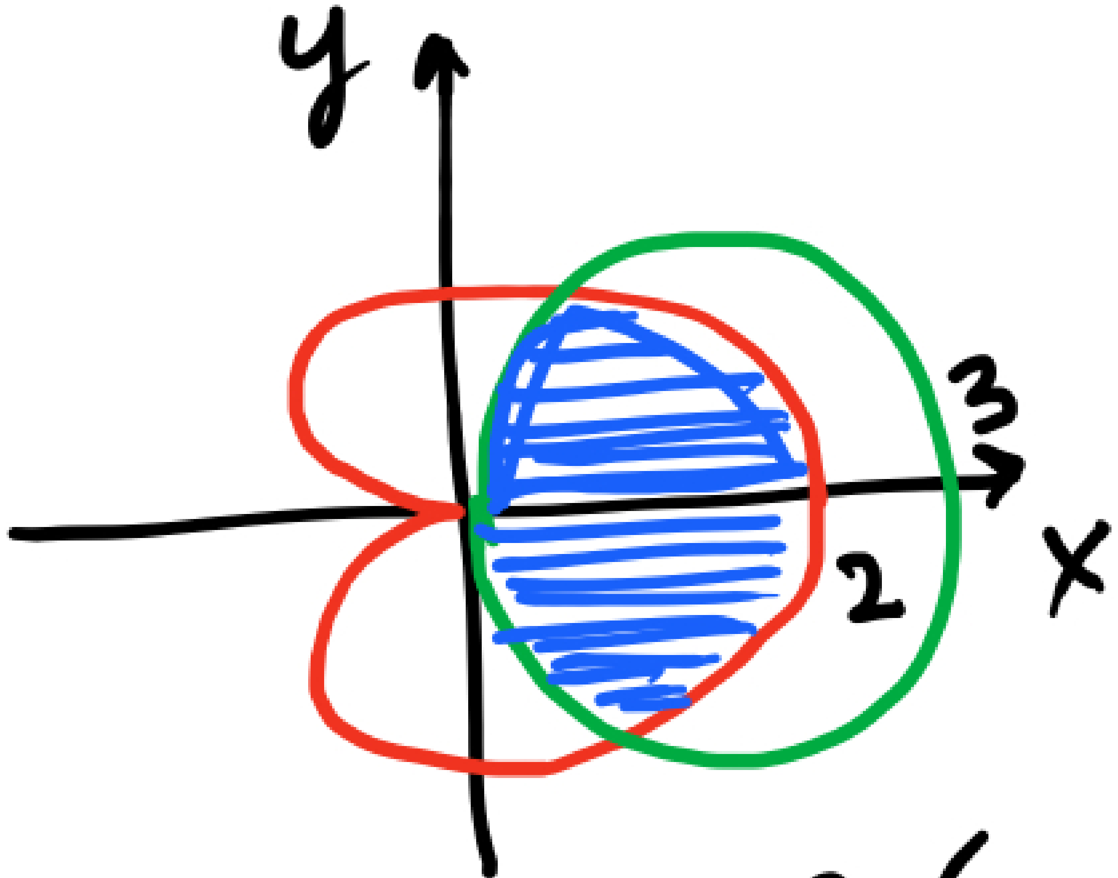
$$A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$$

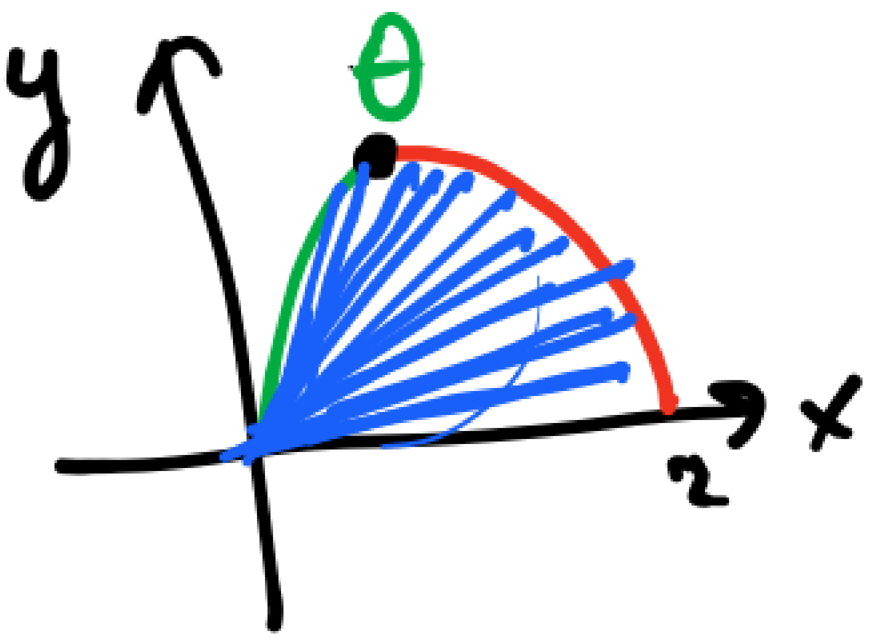
$$= \int_0^{2\pi} \frac{1}{2} \theta^2 \, d\theta =$$

$$= \frac{\theta^3}{6} \Big|_0^{2\pi} = \frac{8\pi^3}{6} =$$

$$= \boxed{\frac{4\pi^3}{3}}$$

(4) Area inside both cardioid $r = 1+\cos\theta$ and the circle $r = 3\cos\theta$





Area =

$$= 2\left(\int_0^{\theta} \frac{1}{2}(1+\cos\theta)^2\,d\theta + \int_{\theta}^{\pi/2} \frac{1}{2}(3\cos\theta)^2\,d\theta \right)$$

intersection:

$$1 + \cos\theta = 3\cos\theta$$

$$2\cos\theta = 1$$

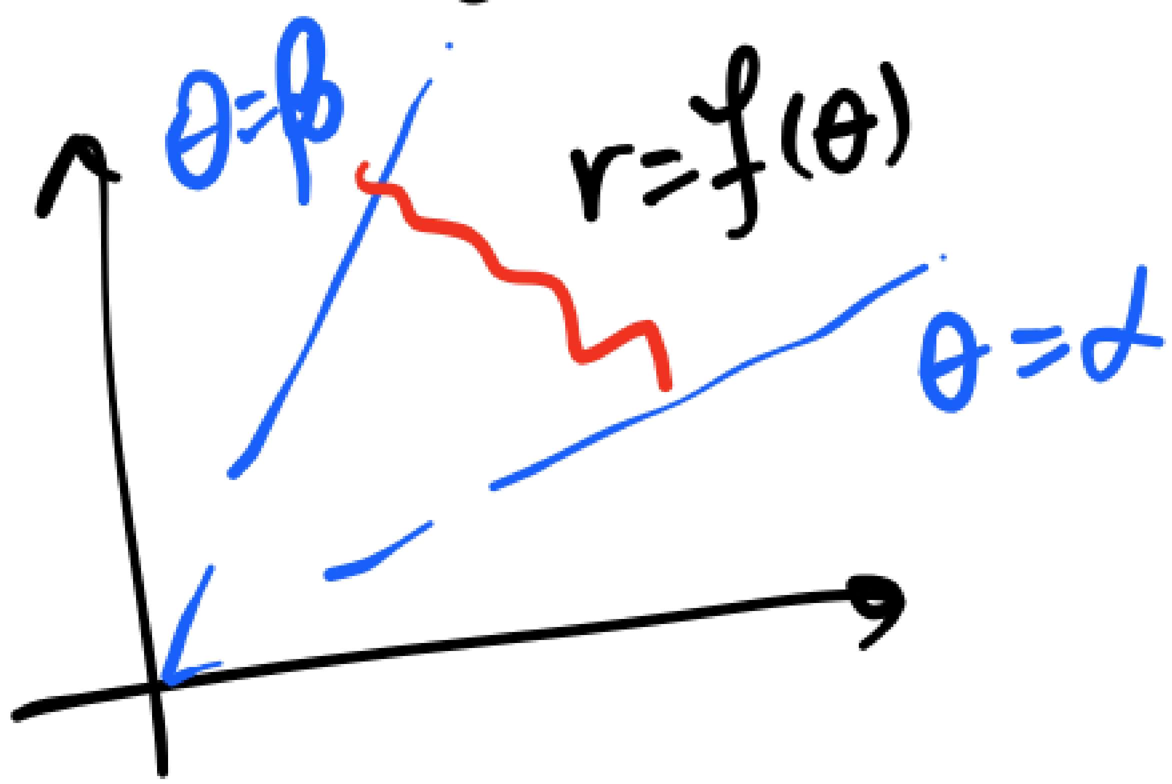
$$\theta = \frac{\pi}{3}.$$

$$A = 2\left(\int_0^{\pi/3} \frac{1}{2}(1+\cos\theta)^2 \, d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(3\cos\theta)^2 \, d\theta \right)$$

check!

$$= \dots = \boxed{\frac{5\pi}{4}}$$

Arc length of polar curves



$s=$ arclength of $r=f(\theta)$ from $\theta=\alpha$ to $\theta=\beta$.

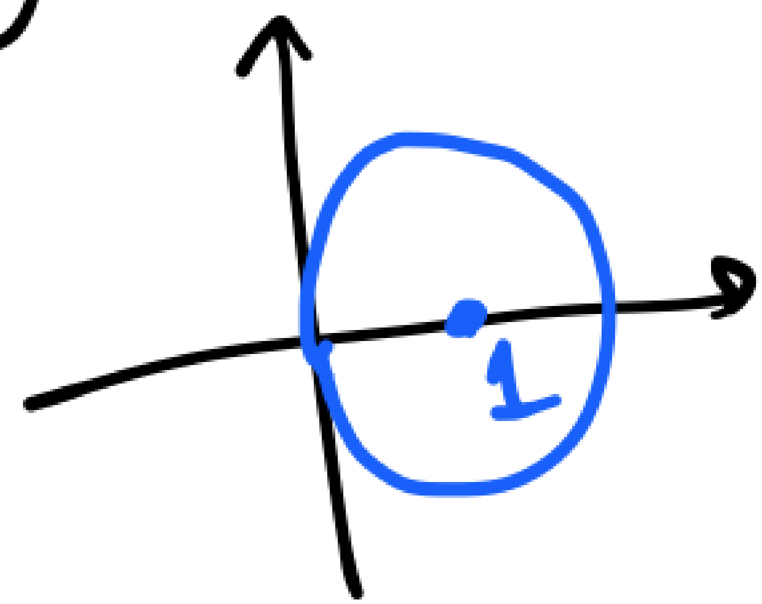
Idea: treat this curve as parametric curve

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases}$$

$$s = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \ldots$$ check

$$= \boxed{\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}.$$

Example 1 $\quad r = 2\cos\theta$



$$s = \int_{0}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta =$$

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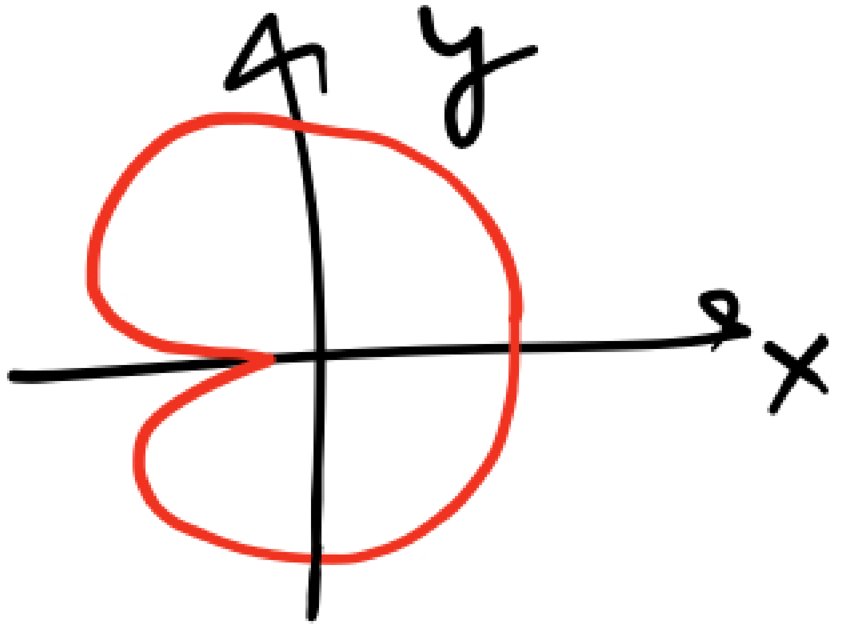
$$= \int_0^{\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta}\, d\theta = \int_0^{\pi} 2\, d\theta = \boxed{2\pi}$$

(2) arclength of entire cardioid $r = 1 + \cos\theta$

$0 \le \theta \le 2\pi$



$$s = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta =$$

$$= \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta =$$

$$= \int_0^{2\pi} \sqrt{1+(\cos^2\theta+\sin^2\theta)+2\cos\theta}\ d\theta =$$

$$\int_0^{2\pi} \sqrt{2+2\cos\theta}\ d\theta = \int_0^{2\pi} \sqrt{4\cos^2\frac{\theta}{2}}\ d\theta =$$

$$= \int_0^{2\pi} 2\left|\cos\frac{\theta}{2}\right| d\theta =$$

$$= \int_0^{\pi} 2\cos\frac{\theta}{2}\, d\theta - \int_{\pi}^{2\pi} 2\cos\frac{\theta}{2}\, d\theta =$$

$$= 4 - (-4) = 8.$$