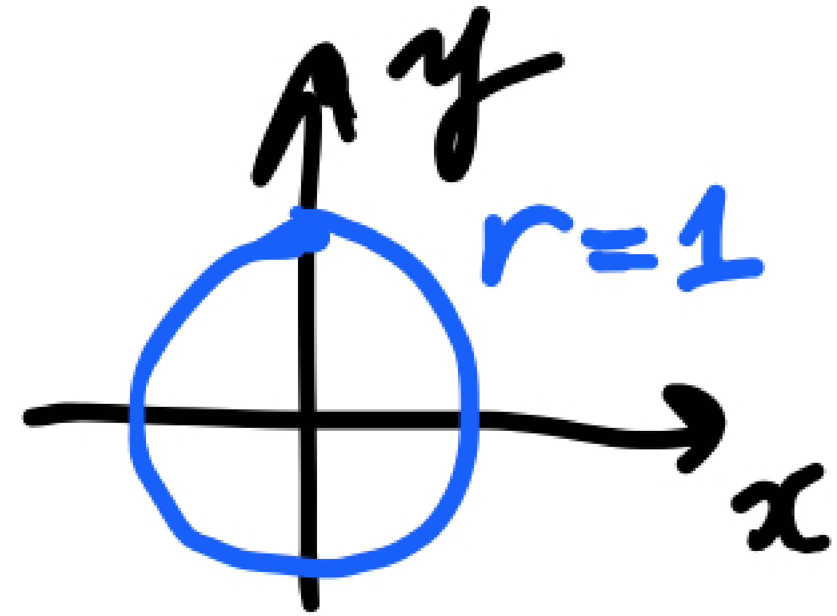
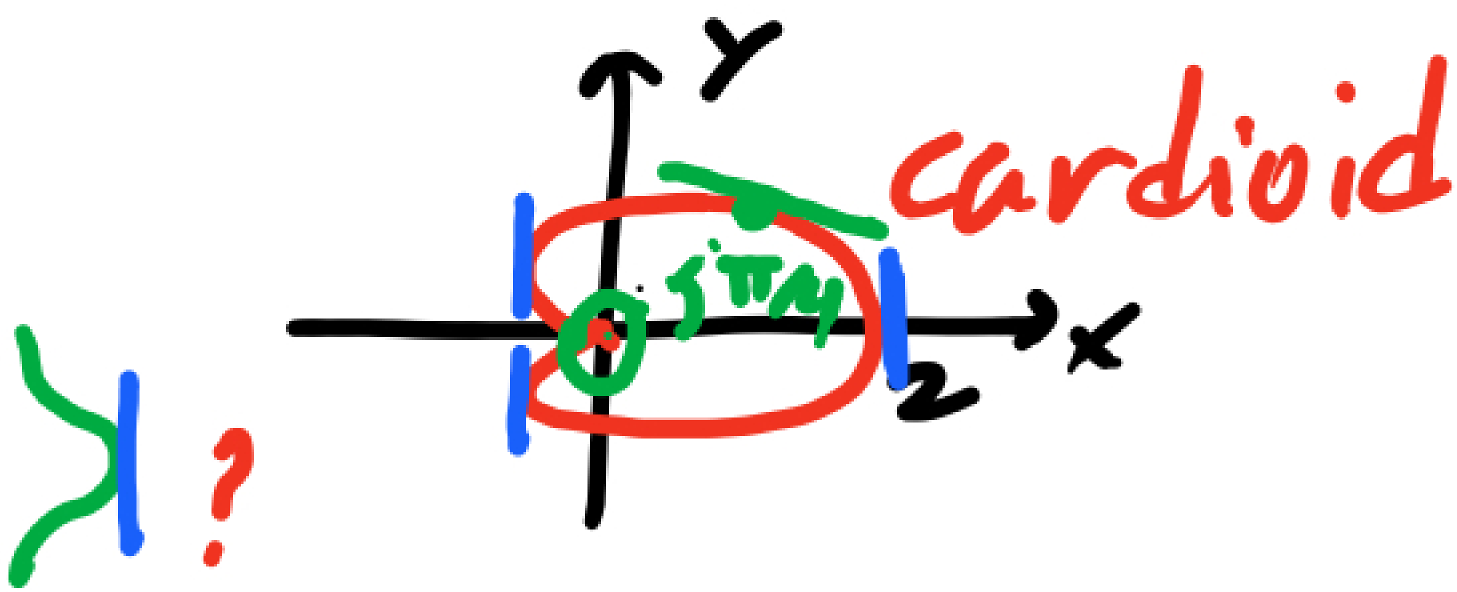
Polar curves

$$r = f(\theta)$$



- spiral
  $r = \theta$
- $r = 2\cos\theta$

- $r = 1 + \cos\theta$



Calculus with polar curves

Idea: treat polar curve as a parametric curve with parameter θ

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases}$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \boxed{\frac{(f(\theta)\sin\theta)'}{(f(\theta)\cos\theta)'}}.$$

Example

(a) Find an $eq^{\underline{n}}$ for the tangent line to $r = 1+\cos\theta$ at $\theta = \pi/4$.

Need $\left.\dfrac{dy}{dx}\right|_{\theta=\frac{\pi}{4}}$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \left.\frac{\big((1+\cos\theta)\sin\theta\big)'}{\big((1+\cos\theta)\cos\theta\big)'}\right|_{\theta=\frac{\pi}{4}} =$$

$$y = \sin\theta + \sin\theta\cos\theta = \sin\theta + \frac{1}{2}\sin 2\theta$$

chain rule

$$\frac{dy}{d\theta} = \cos\theta + \cos 2\theta$$

$$x = \cos\theta + \cos^2\theta =$$

chain rule

$$\frac{dx}{d\theta} = -\sin\theta + 2\cos\theta(-\sin\theta)$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \left.\frac{\cos\theta + \cos 2\theta}{-\sin\theta(1+2\cos\theta)}\right|_{\theta=\pi/4} =$$

$$= \frac{\frac{1}{\cancel{\sqrt{2}}} + 0}{-\frac{1}{\cancel{\sqrt{2}}}(1+\sqrt{2})} = -\frac{1}{1+\sqrt{2}}.$$

<u>Recall</u> $\quad y - y_0 = \underbrace{m}_{\text{slope}}(x - x_0)$

$\underline{(x_0, y_0)}$
the point

$$x_0 = r\cos\theta \Big|_{\theta=\frac{\pi}{4}} = \left(1+\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{1}{2}.$$

$$y_0 = r\sin\theta \Big|_{\theta=\frac{\pi}{4}} = \left(1+\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{1}{2}$$

$$\boxed{y - \left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right) = -\frac{1}{1+\sqrt{2}}\left(x - \left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right)\right)}$$

- equation of the tangent line to cardioid at $\theta = \frac{\pi}{4}$.

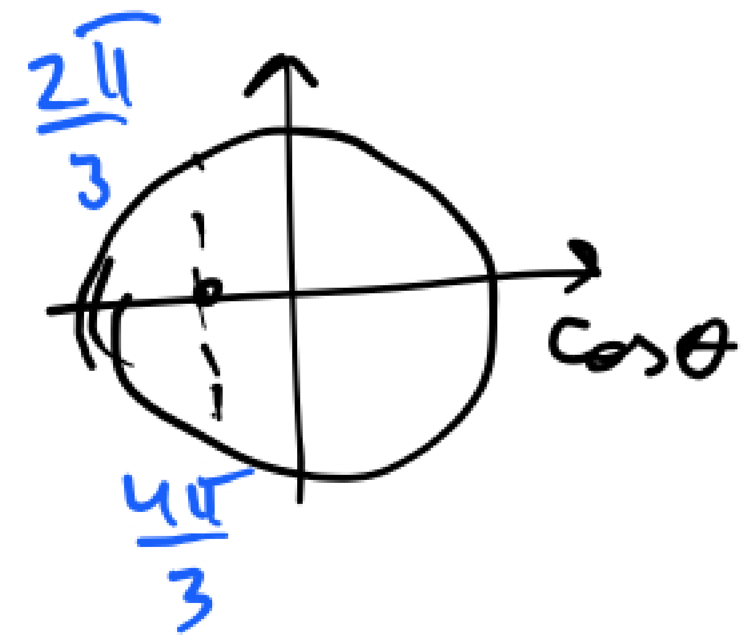
(b) Are there pts where is vertical?

$$-\sin\theta(1+2\cos\theta)=0$$

$$\cos\theta+\cos 2\theta \neq 0$$

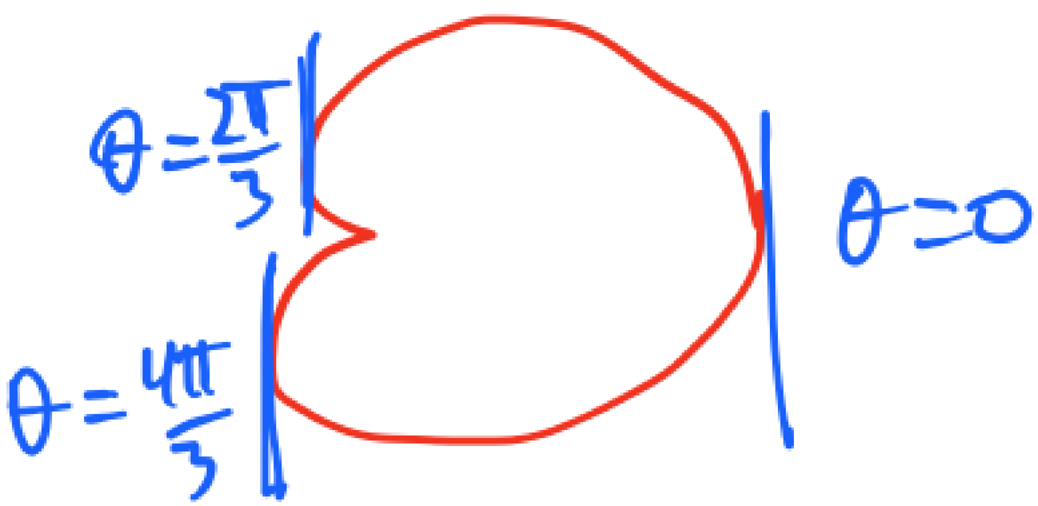
$\sin\theta = 0$ or $1+2\cos\theta = 0$

$\cos\theta = -1/2$



$\theta = 0, \pi$ ✓ ✗

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ ✓ ✓

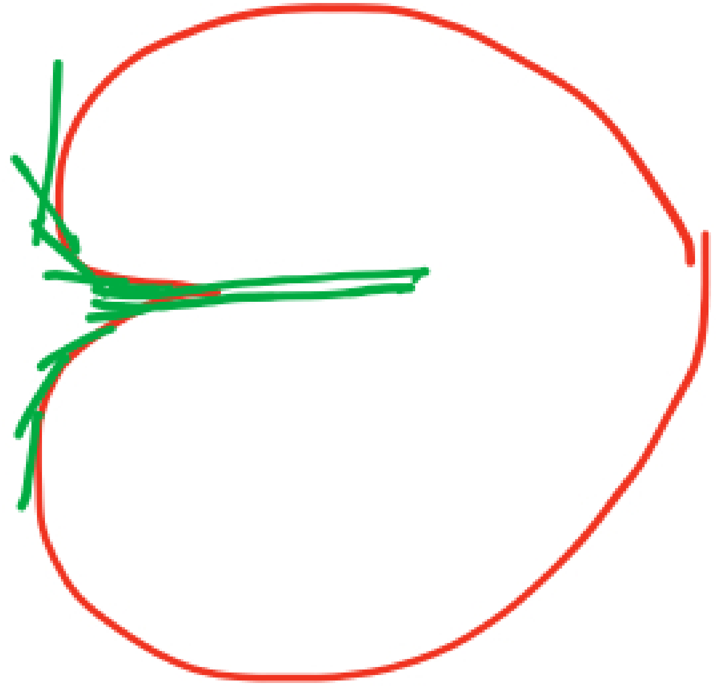


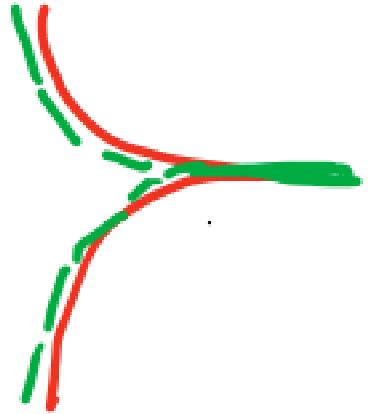
?

What happens at $\theta = \pi$?

In this case "$\frac{dy}{dx} = \frac{0}{0}$".

Look at $\lim_{\theta \to \pi} \frac{dy}{dx} = \underset{\text{check!}}{\ldots} = 0.$

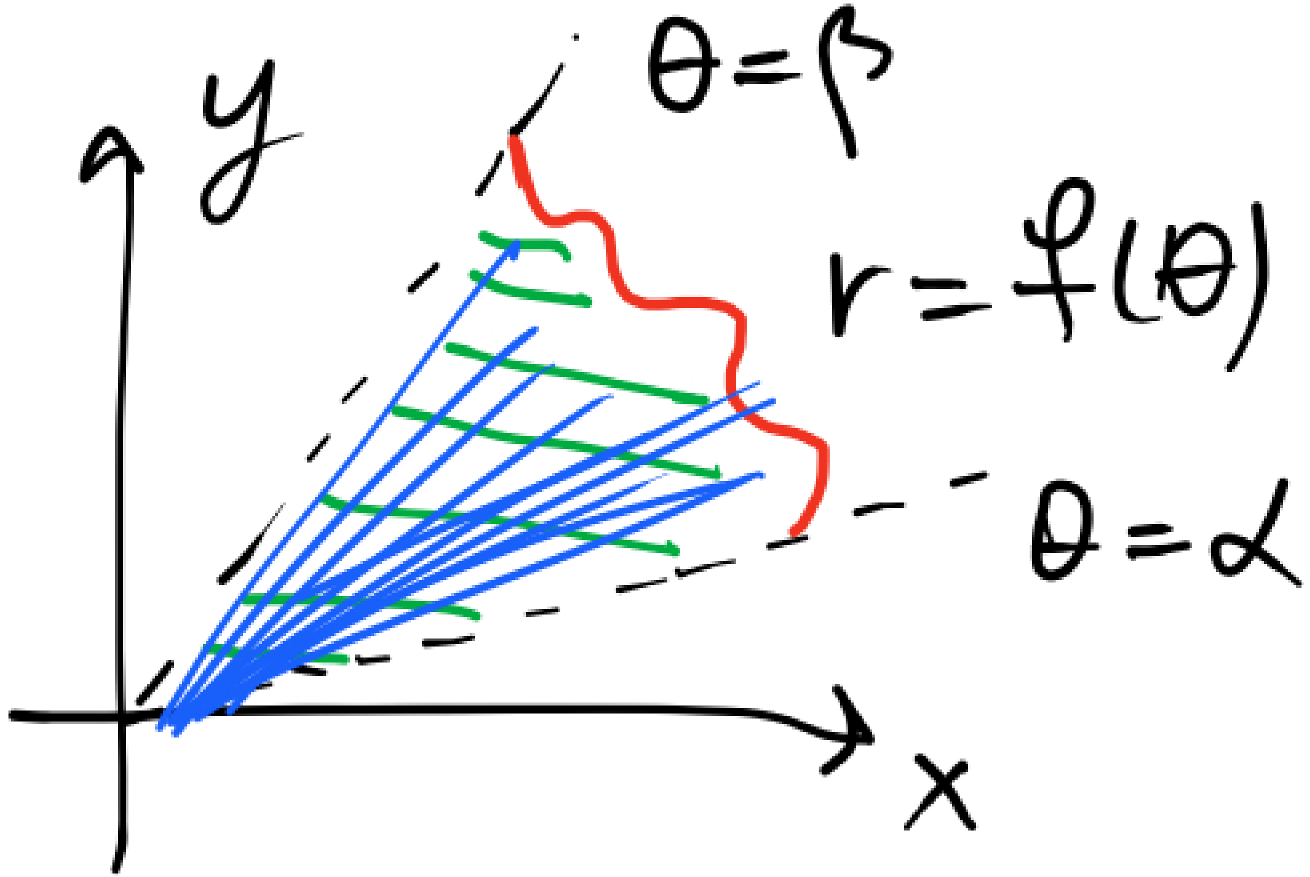




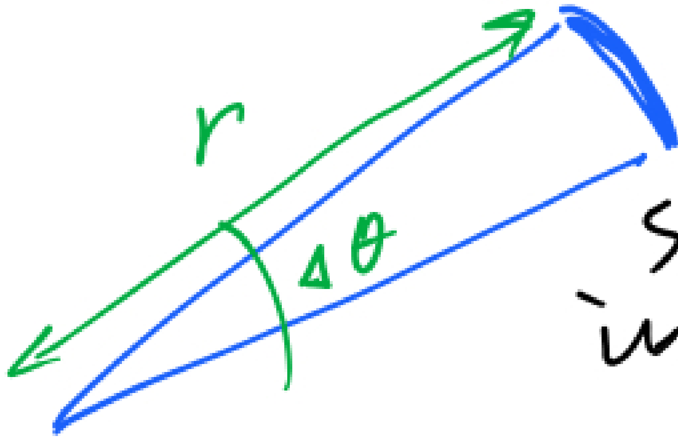
10.4. Areas and Lengths in Polar coordinates.

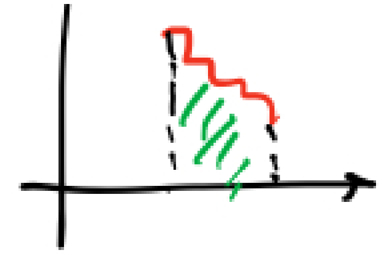
Areas

$\alpha \le \theta \le \beta$



- NOT the same as area under graph and above x-axis





slice the region into pieces that are $\approx$ circular sectors of area ΔA

$$\Delta A = \frac{1}{2} r^2 \Delta\theta$$

Then

$$A \approx \sum \Delta A = \sum \frac{1}{2} r^2 \Delta\theta$$

$$A = \lim_{\Delta\theta \to 0} \left(\sum \frac{1}{2} r^2 \Delta\theta \right) = \boxed{\int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta}$$

area swept by polar curve