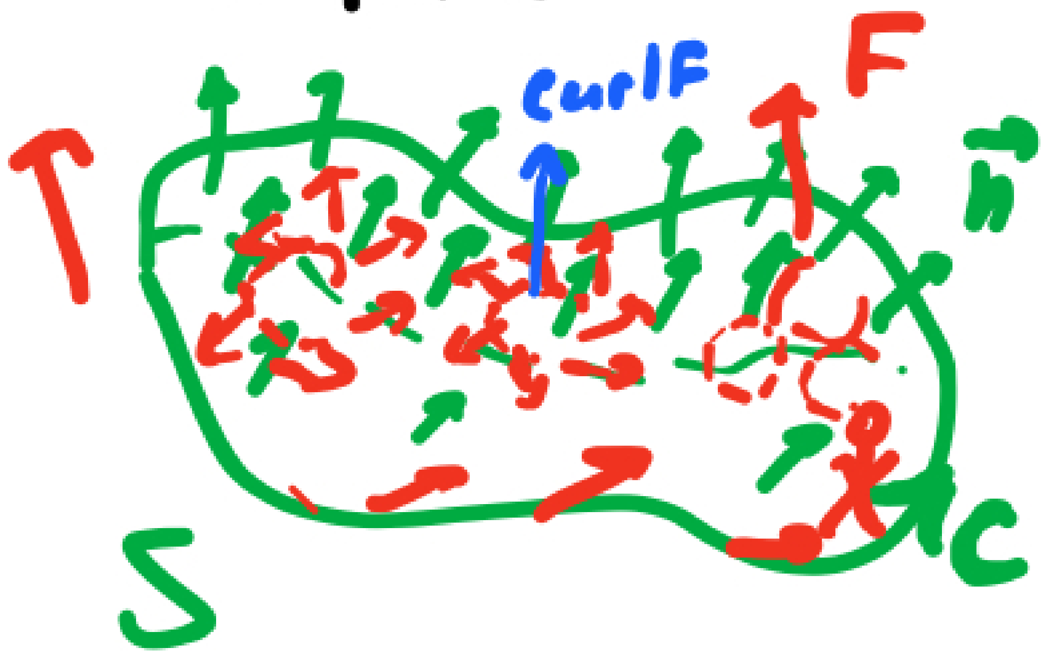
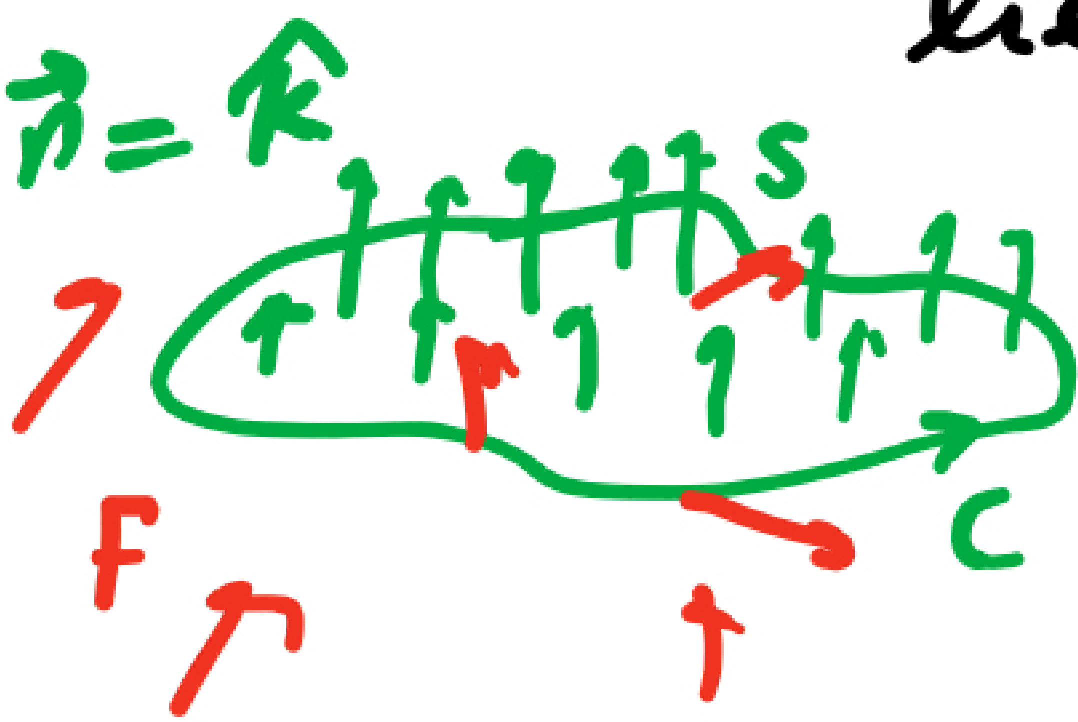
Stokes' theorem



$$\boxed{\int_C F \cdot dr = \iint_S \operatorname{curl} F \cdot d\vec{S}}$$

Says that the line integral of the tangential component of F around C equals to the surface integral of the normal component of curl F.

→ If $S = D$ is a flat surface and lies in xy-plane with upward orientation ($\vec{n} = \hat{k}$) then Stokes' thm says



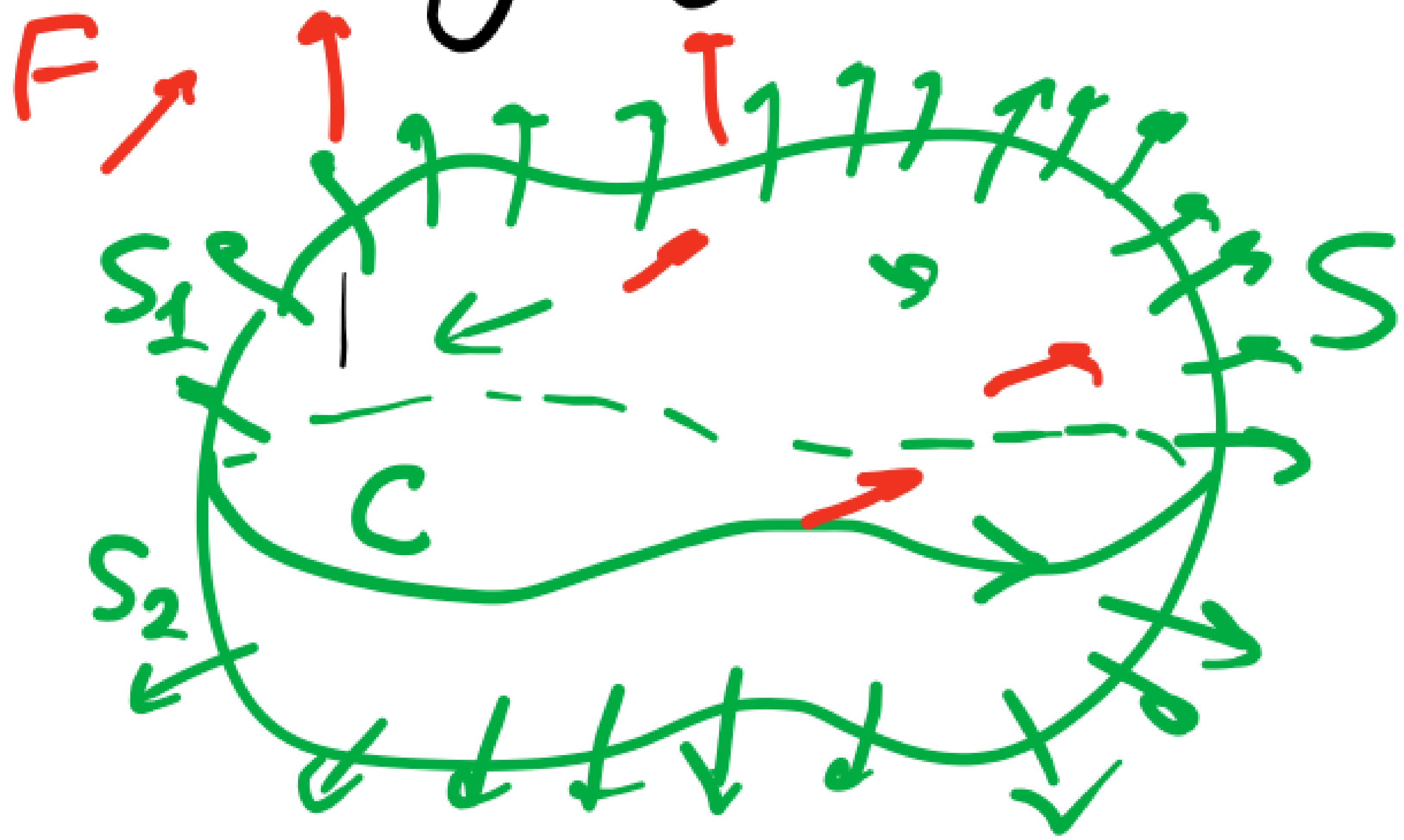
$$\underline{\int F \cdot dr} = \iint_S \operatorname{curl} F \cdot \vec{n} \, dS$$

$$= \iint_S \operatorname{curl} F \cdot \hat{k} \, dS$$

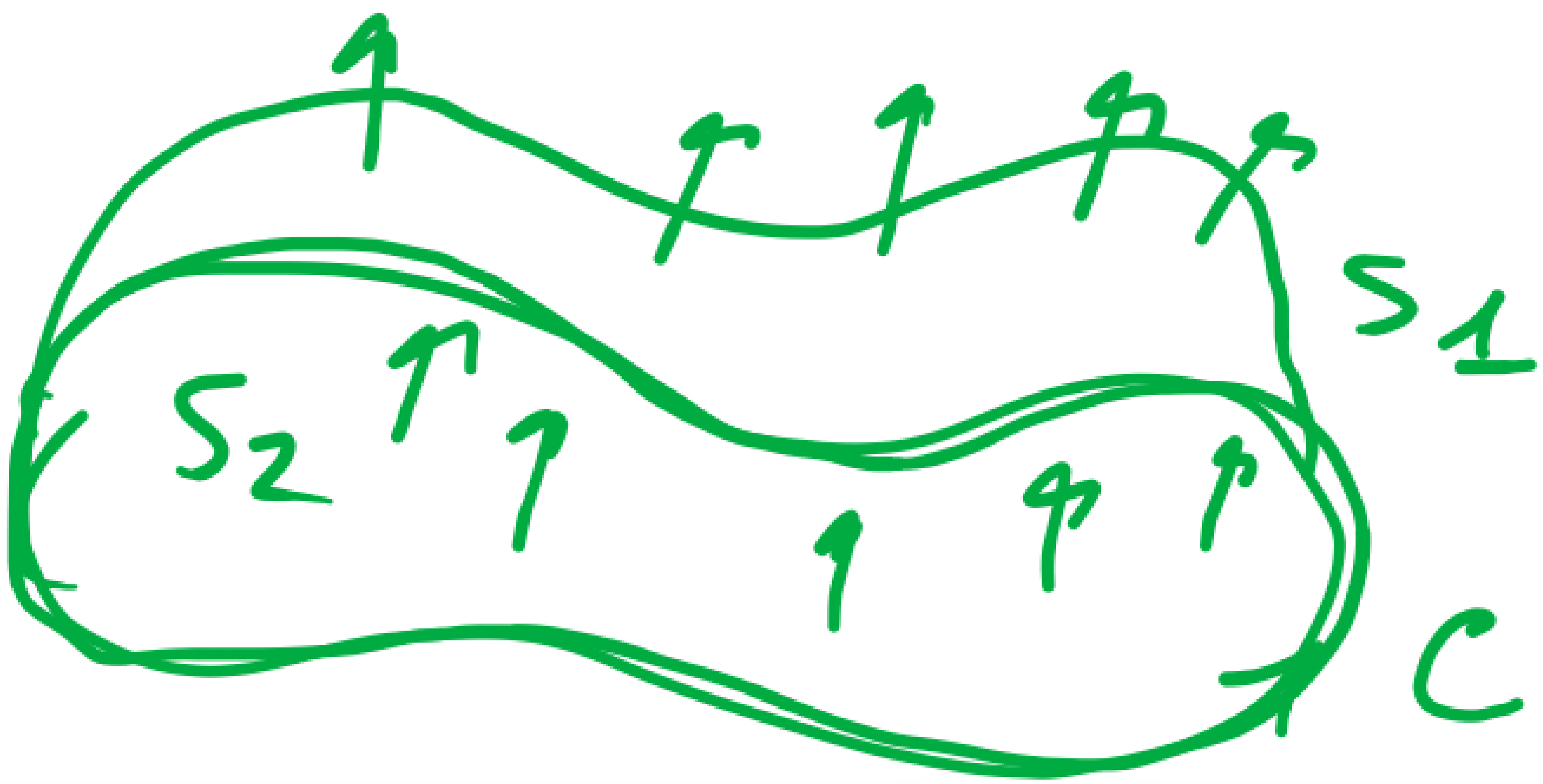
$$= \underline{\iint_D \operatorname{curl} F \cdot \hat{k} \, dA}$$

vector form of Green's thm

Corollary of Stokes' theorem:



- $\int_C F \cdot dr = \iint_{S_1} \operatorname{curl} F \cdot d\vec{S}$
- $\int_C F \cdot dr = -\iint_{S_2} \operatorname{curl} F \cdot d\vec{S}$

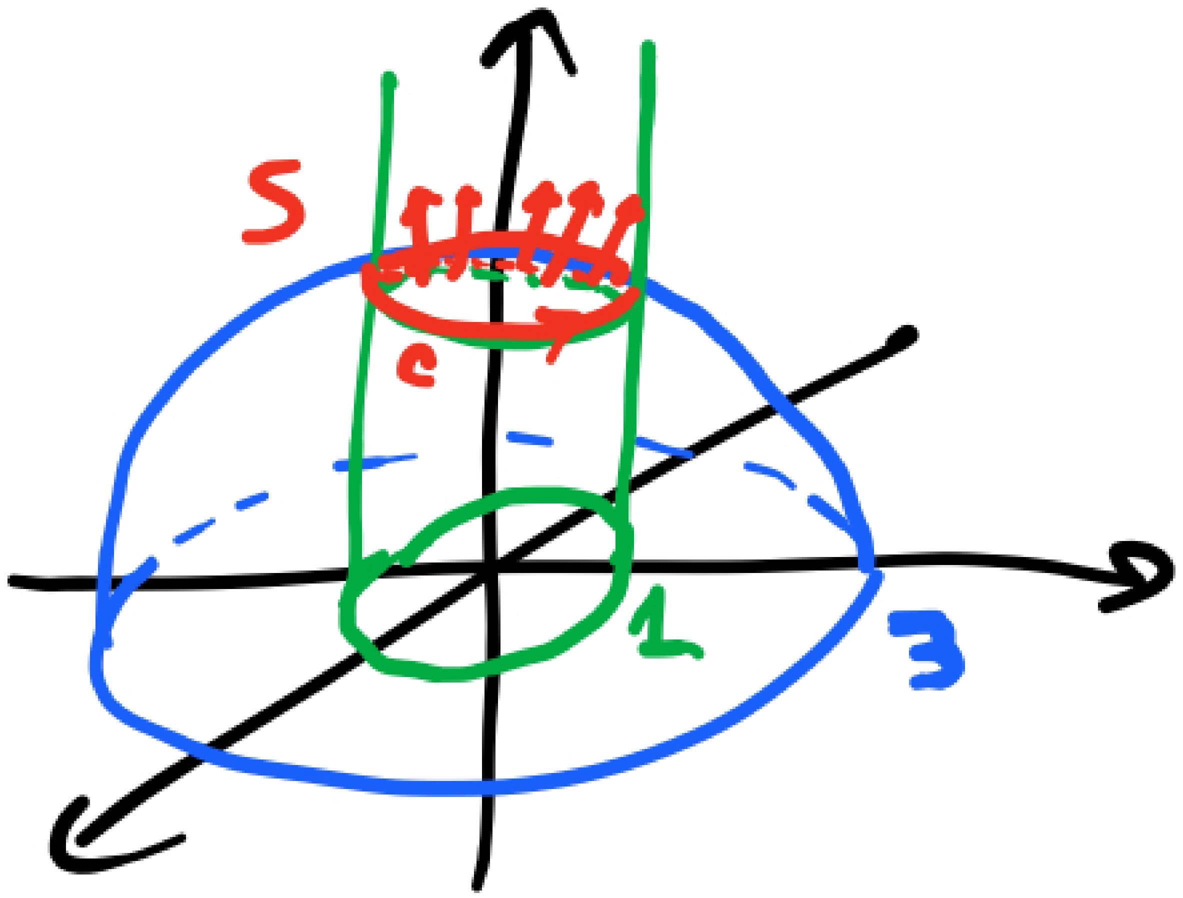


$$\iint_{S_1} \operatorname{curl} F \cdot d\vec{S} = \iint_{S_2} \operatorname{curl} F \cdot d\vec{S}.$$

Example Find $\iint_S \text{curl}\, F \cdot d\vec{S}$, where

$F = x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k}$ and S

is:



One appoach:

By Stokes' theorem

$$\iint_S \text{curl}\, F \cdot d\vec{S} = \int_C F \cdot dr$$

$\cancel{r(t) =}$ Boundary curve C: $x^2 + y^2 = 1$

$x^2 + y^2 + z^2 = 9$

$$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 2\sqrt{2}\,\hat{k} \qquad z^2 = 8$$

$$0 \le t \le 2\pi \qquad z = +2\sqrt{2}$$

$$r'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + 0\,\hat{k}$$

$$F(r(t)) = \cos^2 t\,\hat{i} + \sin^2 t\,\hat{j} + 8\,\hat{k}$$

$$\int_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t)\,dt =$$

$$\int_0^{2\pi} \left(-\sin t \cos^2 t + \sin^2 t \cos t\right) dt =$$

$$= \frac{\cos^3 t}{3} + \frac{\sin^3 t}{3} \Big|_0^{2\pi} = 0.$$

Another approach:

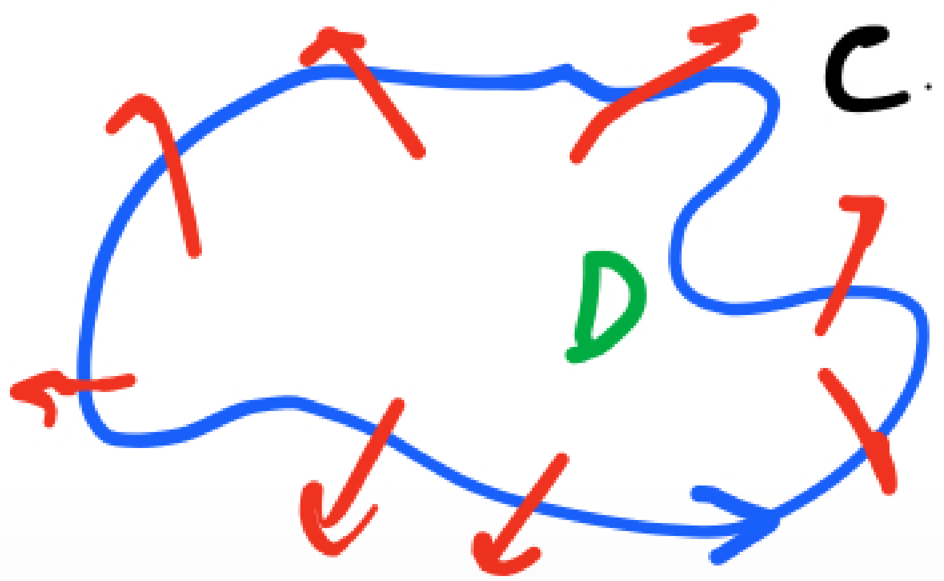
By Stokes' thm,
$$\iint_{S} \operatorname{curl} F \cdot d\vec{S} = \iint_{S'} \operatorname{curl} F \cdot d\vec{S}$$

where S' is the disk whose boundary is C. (with upward orientation). However, $\operatorname{curl} F = 0$.

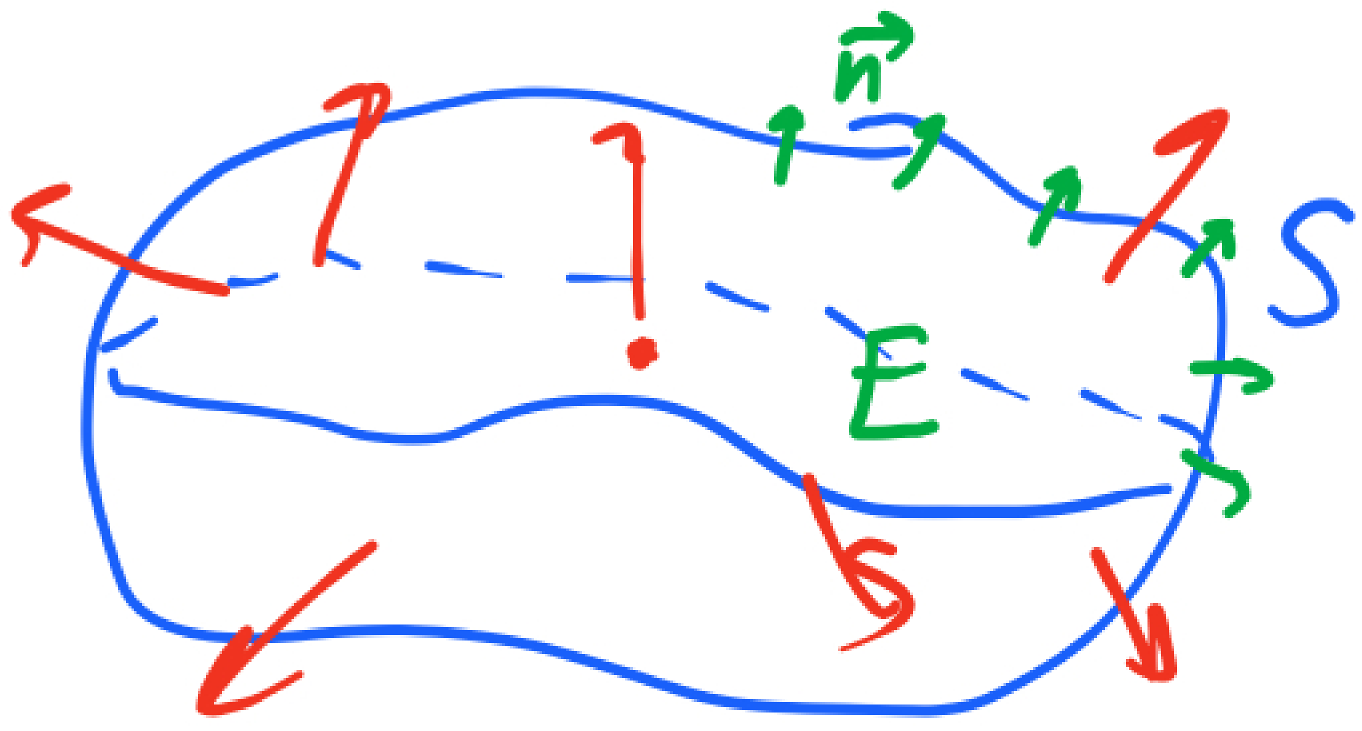
Divergence theorem (16.9).

Recall another vector form of Green's thm:

$$\int_C F \cdot \vec{n}\, ds = \iint_D \operatorname{div} F\, dA$$



If we would extend it to $\mathbb{R}^3$ we would write:



$$\iint_S F \cdot \vec{n} \, dS = \iiint_E \operatorname{div} F \, dV$$

Divergence theorem

Example (from yesterday)

$F = \langle z, x, y \rangle$

$\operatorname{div} F = 0.$

$\text{flux} = \iiint 0 \, dV = 0.$