Last time:

- surface integral of a vector field

$$\rightarrow \iint_S F \cdot d\vec{S} = \iint_S F \cdot \vec{n}\, dS$$

$$\rightarrow \iint_S F \cdot d\vec{S} = \iint_D F \cdot (r_u \times r_v)\, dA$$

Case of the graph $z = f(x,y)$:

$\rightarrow$ $F \cdot (r_u \times r_v) = ?$

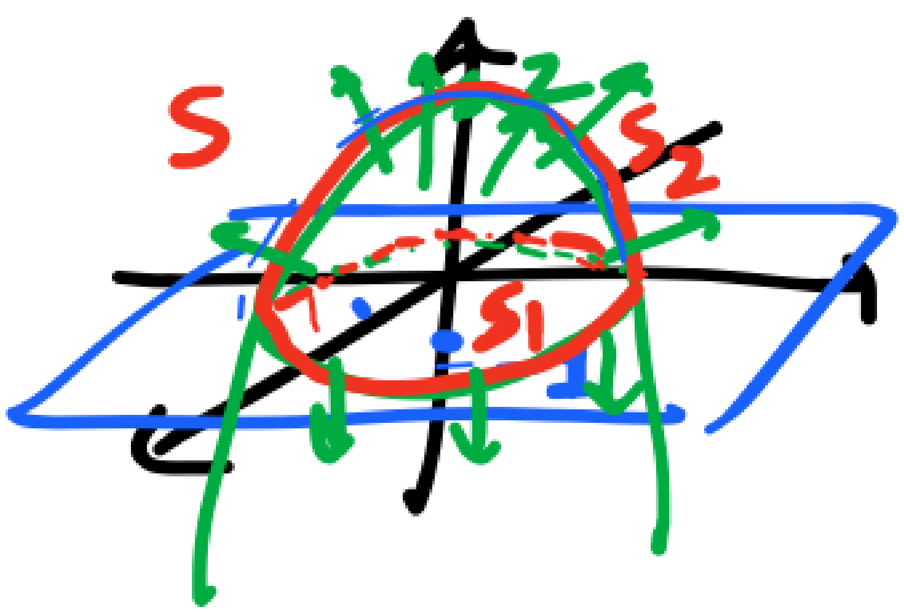
$$\begin{cases} x = x \\ y = y \\ z = f(x,y) \\ (x,y) \in D \end{cases}$$

$$r_x = 1\hat{\imath} + 0\hat{\jmath} + f_x\hat{k}$$
$$r_y = 0\hat{\imath} + 1\hat{\jmath} + f_y\hat{k}$$
$$r_x \times r_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} =$$
$$= -f_x\hat{\imath} - f_y\hat{\jmath} + \hat{k}$$

Thus
$$\iint_S F \cdot d\vec{S} = \iint_D (P\hat{\imath} + Q\hat{\jmath} + R\hat{k}) \cdot (-f_x\hat{\imath} - f_y\hat{\jmath} + \hat{k})\, dA =$$
$$= \iint_D (-P \cdot f_x - Q \cdot f_y + R)\, dA$$

__Example__ Find $\iint_S F \cdot d\vec{S}$ where

$F = z\hat{\imath} + x\hat{\jmath} + y\hat{k}$ and S is the boundary of the solid E enclosed by plane $z = -1$ and paraboloid $z = 2 - x^2 - y^2$, positively oriented.



$-1 = 2 - x^2 - y^2$

$\to x^2 + y^2 = 3, \quad z = -1.$

- flux across the bottom disk S_1

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$$\iint_{S_1} \bar{F} \cdot d\vec{S} = \iint_{S_1} F \cdot \vec{n} \, dS =$$

$$= \iint_{S_1} (z\hat{i} + x\hat{j} + y\hat{k}) \cdot (-\vec{k}) \, dS = \iint_{S_1} -y \, dS =$$

$$= \iint_D -y \, dA = \int_0^{2\pi} \int_0^{\sqrt{3}} -(r\sin\theta) \, r \, dr \, d\theta =$$

$$= -\int_0^{\sqrt{3}} r^2 dr \cdot \int_0^{2\pi} \sin\theta \, d\theta = -\frac{r^3}{3}\Big|_0^{\sqrt{3}} \cdot \underbrace{-\cos\theta\Big|_0^{2\pi}} =$$

$$= 0.$$

- flux across the paraboloid piece S_2:

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_D \left(-P \cdot f_x - Q \cdot f_y + R\right) dA =$$

$$= \iint_D \left(-z \cdot (-2x) - x \cdot (-2y) + y\right) dA =$$

$$= \iint_D \left(2x\underset{\substack{\downarrow \\ (2-x^2-y^2)}}{z} + 2xy + y\right) dA = \int_0^{2\pi}\int_0^{\sqrt{3}} \left(2r\cos\theta\,(2-r^2) + 2r^2\sin\theta\cos\theta + r\sin\theta\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \left(\left(4r - 2r^3\right)\cos\theta + 2r^2\sin\theta\cos\theta + r\sin\theta\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \left(\left(4r^2 - 2r^4\right)\cos\theta + \cancel{2}r^3 \sin 2\theta + r^2 \sin\theta \right) dr\, d\theta$$

$$= \int_0^{2\pi} \left(\left(\frac{4r^3}{3} - \frac{2r^5}{5} \right)\cos\theta + \frac{r^4}{4}\sin 2\theta + \frac{r^3}{3}\sin\theta \right)\Bigg|_0^{\sqrt{3}} d\theta =$$

$$= \int_0^{2\pi} \left(\left(4\sqrt{3} - \frac{18\sqrt{3}}{8} \right)\cos\theta + \frac{9}{4}\sin 2\theta + \sqrt{3}\sin\theta \right) d\theta =$$

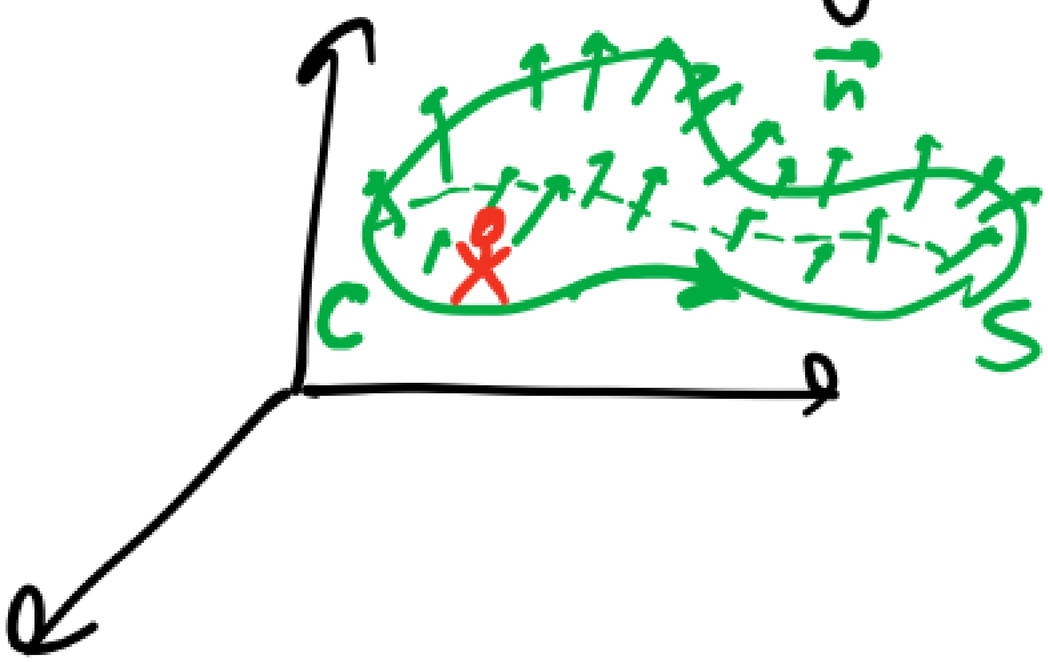
$$= \ldots = 0.$$

Thus $\displaystyle\iint_S F \cdot d\vec{S} = \iint_{S_1} F \cdot d\vec{S} + \iint_{S_2} F \cdot d\vec{S} =$

$$= 0 + 0 = 0$$

16.8. Stokes' theorem

a generalization of Green's theorem



Given:

- oriented smooth surface S in $\mathbb{R}^3$ with unit vector $\vec{n}$
- boundary curve C with induced positive orientation

means that if you walk in this positive direction along C with your head pointing in the direction of $\vec{n}$, then the surface is on your left.

Stokes' Theorem

Suppose S and C are as above.
Let F be a vector field whose components have continuous partial derivatives.
and

Then

$$\boxed{\int_C F \cdot dr = \iint_S \operatorname{curl} F \cdot d\vec{S}}$$