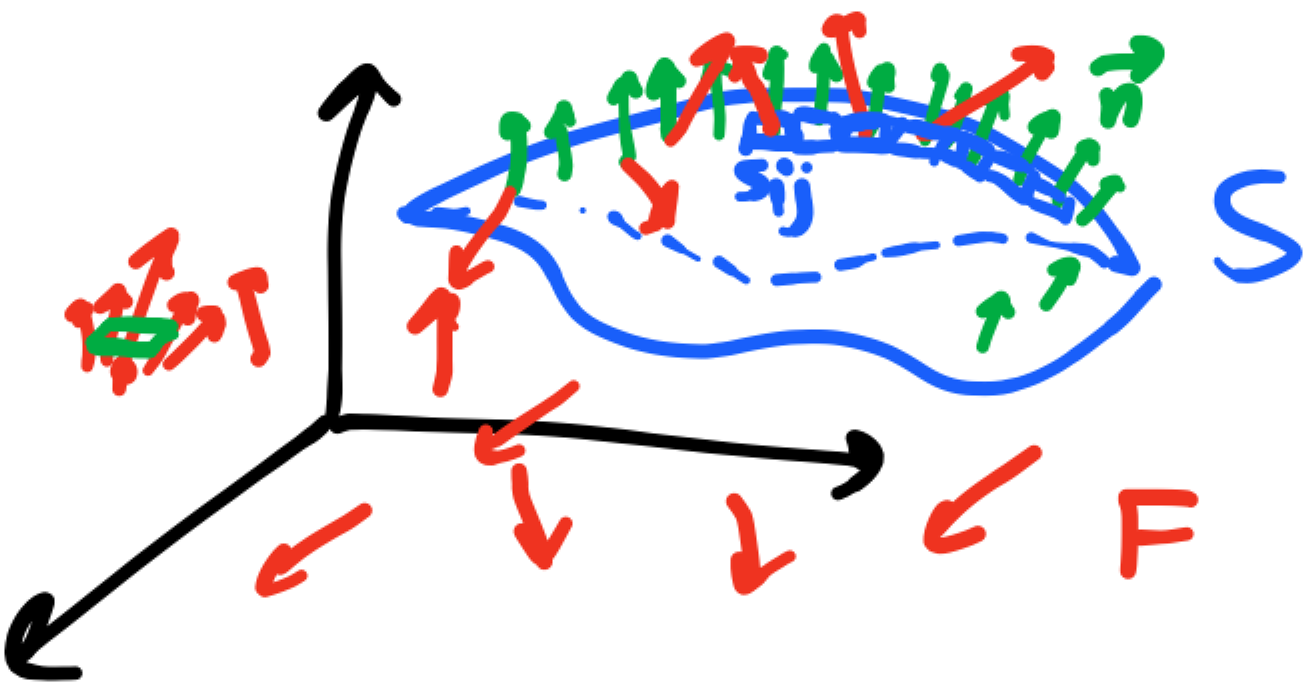
Surface integral of a vector field

Set up: S – smooth oriented surface in $\mathbb{R}^3$ and



F is a continuous vector field in $\mathbb{R}^3$.

Interpretation: suppose that a fluid is flowing through S ("fishing net") and sps $v(x,y,z)$ - velocity vector field of the fluid and $\rho(x,y,z)$ - density function.

- the rate of the flow (mass per unit time) per unit area is ρv vector field (F)

- divide S into small nearly planar pieces, then mass per unit time crossing S_{ij} in the direction of $\vec{n}$ is approx $(\rho v \cdot \vec{n}) A(S_{ij})$

• in the limit as S_{ij} get smaller

$$\iint_S \rho v \cdot \vec{n}\, dS$$ – the rate of the flow through S.

If we replace $F = \rho v$: $$\iint_S \underbrace{F \cdot \vec{n}}_{\text{function}}\, dS$$

$$\boxed{\iint_S F \cdot \vec{n}\, dS}$$ – surface integral of F over S.

Another notation: $$\boxed{\iint_S \underline{F \cdot dS}}$$

Remark:

this integral is also called flux of F across S.

compare with

$$\int_C F \cdot dr$$

Formula: If S is given by

$r(u,v)$, $(u,v) \in D$ (parametrization)

$$\iint_S F \cdot dS = \iint_S F \cdot \frac{r_u \times r_v}{\|r_u \times r_v\|} \, dS =$$

$$= \iint_D F \cdot \frac{r_u \times r_v}{\cancel{\|r_u \times r_v\|}} \, \cancel{\|r_u \times r_v\|} \, dA =$$

$$= \iint_D F \cdot (r_u \times r_v) \, dA$$

$$\rightarrow \boxed{\iint_S F \cdot dS = \iint_D F \cdot (r_u \times r_v) \, dA}$$

compare with

$$\int_C F \cdot dr = \int_a^b F \cdot r' \, dt.$$

__Example__ Find the flux of

$F = x\hat{i} + y\hat{j} + z\hat{k}$ across the unit sphere centered at the origin. (positively oriented)

$$r(\varphi,\theta) = \sin\varphi\cos\theta\,\hat{i} + \sin\varphi\sin\theta\,\hat{j} + \cos\varphi\,\hat{k}$$

$$(\varphi,\theta) \in [0,\pi] \times [0,2\pi].$$

$$r_\varphi \times r_\theta = \sin^2\varphi\cos\theta\,\hat{i} + \sin^2\varphi\sin\theta\,\hat{j} + \sin\varphi\cos\varphi\,\hat{k}.$$

@ $\varphi = \pi/2,\ \theta = 0$: $(1,0,0)$ ✓ orientation is correct.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(\varphi, \theta)) \cdot (\mathbf{r}_\varphi \times \mathbf{r}_\theta)\, dA =$$

$$= \int_0^{2\pi} \int_0^{\pi} \Big(\underbrace{\underbrace{\sin^3\varphi \cos^2\theta + \sin^3\varphi \sin^2\theta}_{\sin^3\varphi} + \sin\varphi \cos^2\varphi}_{\sin\varphi(\sin^2\varphi + \cos^2\varphi) = \sin\varphi} \Big)\, d\varphi\, d\theta$$

$$= \boxed{4\pi}.$$