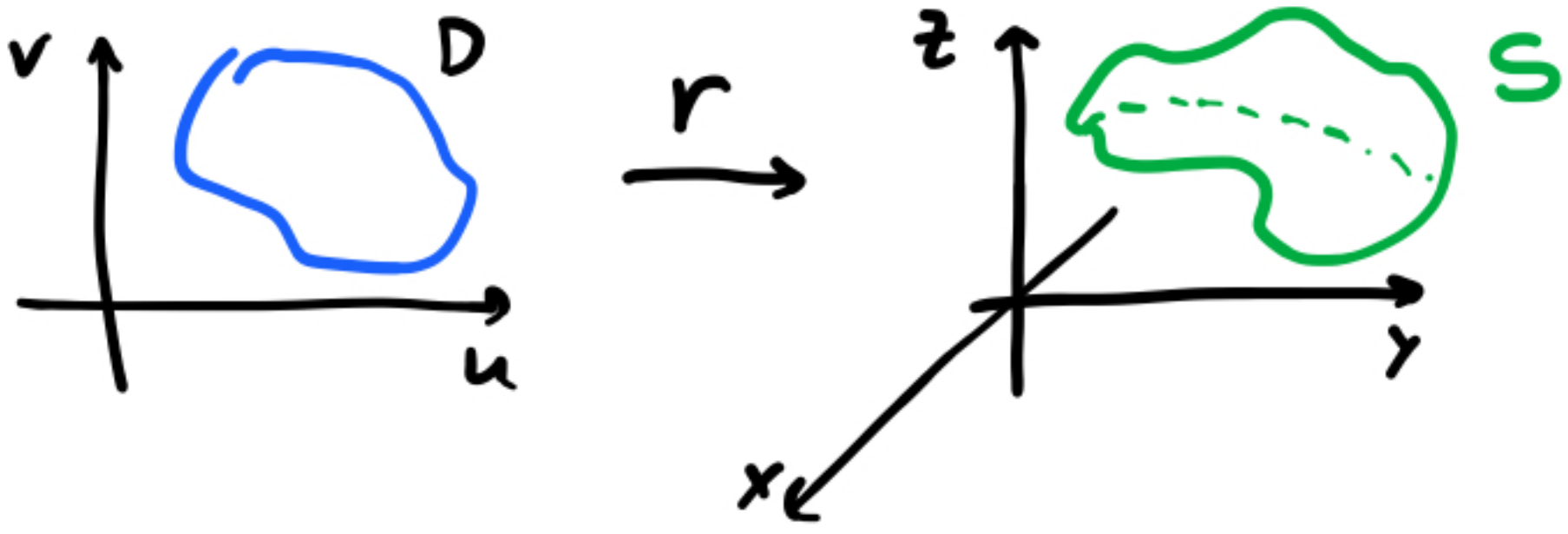
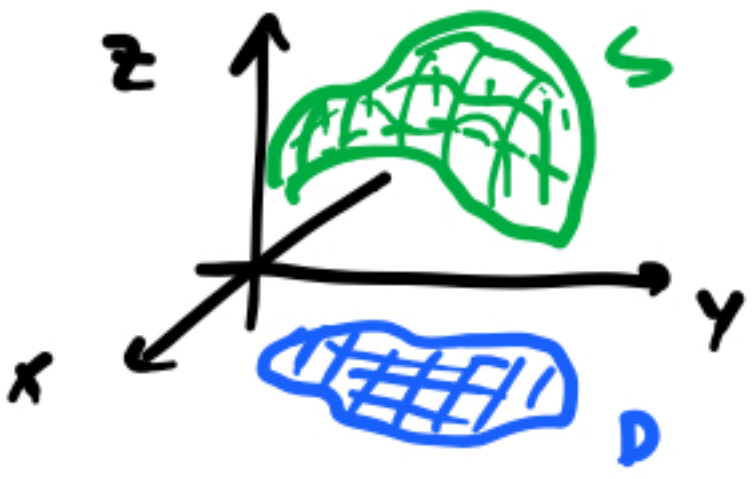
Parametric surfaces



Example $z = f(x,y)$



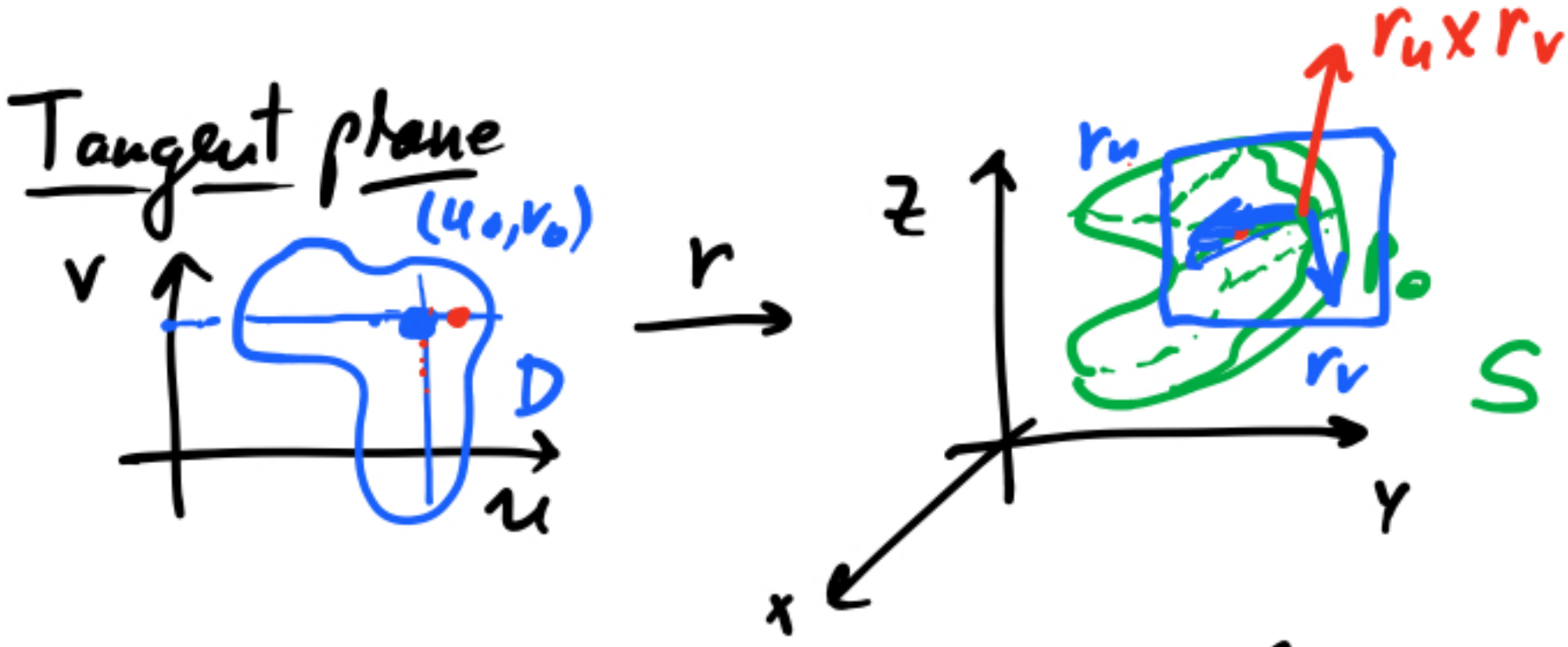
So we can parametrize surface S which is the graph of $z = f(x,y)$:

$$\begin{cases} x = x \\ y = y \\ z = f(x,y) \end{cases} \qquad (x,y) \in D$$

in vector form:

$$r(x,y) = x\hat{\imath} + y\hat{\jmath} + f(x,y)\hat{k}$$
$$(x,y) \in D .$$

Tangent plane



$$r(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}, \quad (u,v) \in D$$

tangent vectors at P_0 can be obtained from partial derivatives of r:

$$\frac{r(u_0+\Delta u, v_0) - r(u_0, v_0)}{\Delta u} \overset{\text{by definition}}{=} r_u =$$

$$\lim_{\Delta u \to 0} \overset{\text{by definition}}{=} \frac{\partial x(u,v)}{\partial u}\hat{i} + \frac{\partial y(u,v)}{\partial u}\hat{j} + \frac{\partial z(u,v)}{\partial u}\hat{k}$$

Similarly, r_v at (u_0, v_0) is also a tangent vector.

- If $r_u \times r_v \neq 0$ the surface is called **smooth**

- For a smooth surface, the tangent plane is the plane that contains r_u and r_v, and $r_u \times r_v$ is a normal vector to the tangent plane

Example Find tangent plane to

$x = uv$
$y = v^2$ at $(1,1,5)$.
$z = 2u + 3v$

$$r_u = \frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j} + \frac{\partial z}{\partial u}\hat{k} = v\hat{i} + 0\hat{j} + 2\hat{k}$$

$$r_v = \frac{\partial x}{\partial v}\hat{i} + \frac{\partial y}{\partial v}\hat{j} + \frac{\partial z}{\partial v}\hat{k} = u\hat{i} + 2v\hat{j} + 3\hat{k}$$

$$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v & 0 & 2 \\ u & 2v & 3 \end{vmatrix} = (-4v)\hat{i} + (2u - 3v)\hat{j} + (2v^2)\hat{k}$$

Point $(1,1,5)$ corresponds to $u=1, v=1$.

So $\vec{n} = -4\hat{\imath} - \hat{\jmath} + 2\hat{k}$.

Equation of the tangent plane:

$$-4(x-1) - (y-1) + 2(z-5) = 0.$$

$$\Leftrightarrow 4x + y - 2z + 5 = 0.$$

Surface area

r

S

Area $(S) = ?$

if S is covered by r just once as (u,v) ranges throughout D, then

$$A(S) = \iint_D \| r_u \times r_v \| \, du\,dv$$

Exercise Find surface area of a sphere of radius $a > 0$.

$(4\pi a^2)$