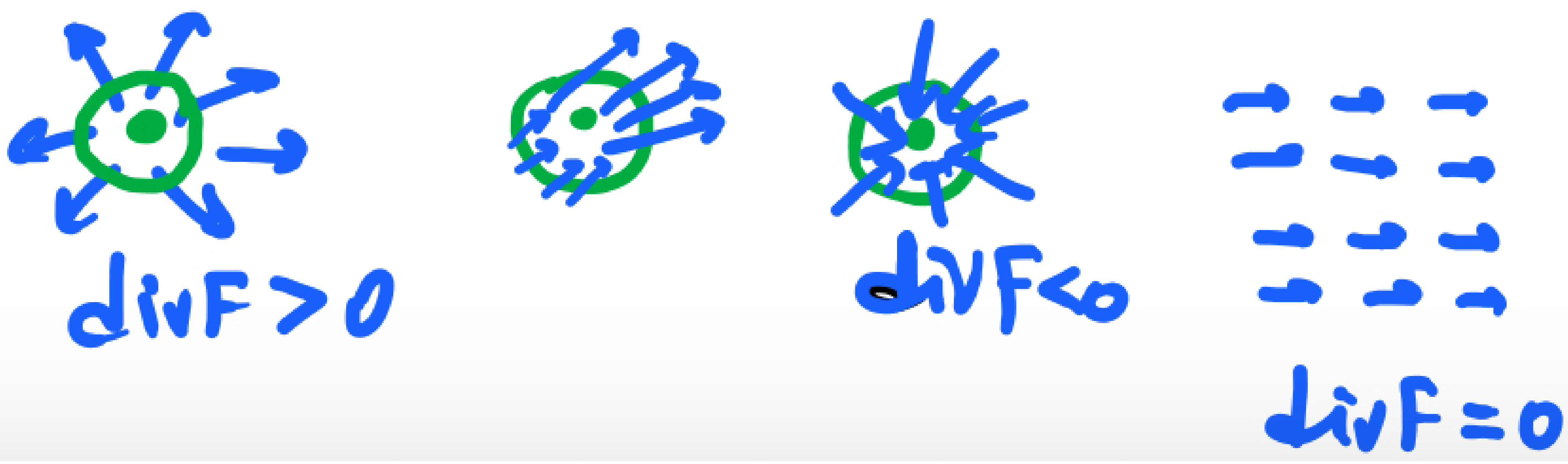
Last time:

- Divergence - operation on vector fields,
  result - function
- $\operatorname{div} F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ (in 2^d)

  $\operatorname{div} F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$ (in 3^d)
- 

Example $F = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$.

$\operatorname{div} F = 2x + 2y + 2z$.

Theorem $\operatorname{div}(\underbrace{\operatorname{curl} F}_{\text{vector field}}) = \underbrace{0}_{\text{identically zero function}}$.

if F is suff. nice
(P,Q,R have continuous second-order partial derivatives) $[P,Q,R \in C^2]$
(compare it with $\operatorname{curl}(\nabla f) = \vec{0}$)

$$\operatorname{div}(\operatorname{curl} F) = \nabla \bullet (\nabla \times F) = \operatorname{div}\left(\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) =$$

$$= \cancel{\frac{\partial^2 R}{\partial x \partial y}} - \cancel{\frac{\partial^2 Q}{\partial x \partial z}} + \cancel{\frac{\partial^2 P}{\partial y \partial z}} - \cancel{\frac{\partial^2 R}{\partial y \partial x}} + \cancel{\frac{\partial^2 Q}{\partial z \partial x}} - \cancel{\frac{\partial^2 P}{\partial z \partial y}} =$$

$$= 0.$$

Vector forms of Green's theorem

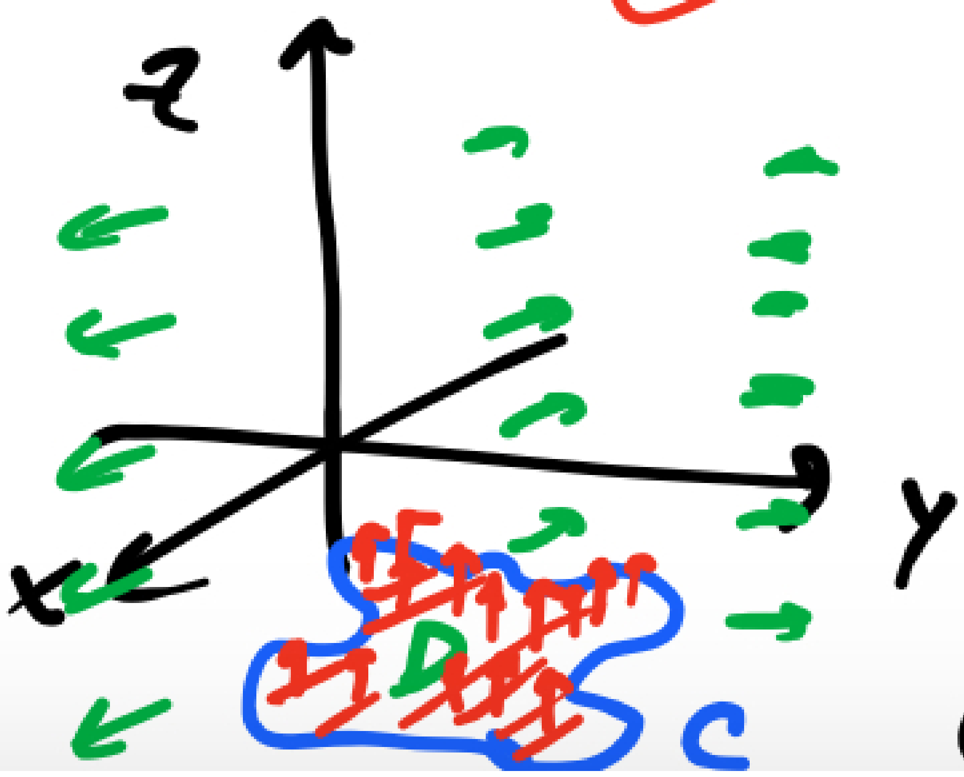
(baby cases of Stokes' thm and Divergence thm)

Recall,

if $F = P\hat{\imath} + Q\hat{\jmath}$

then $\int_C F \cdot dr = \int_C P\,dx + Q\,dy$

Regard this picture living in 3^d.



by extending $F = P\hat{\imath} + Q\hat{\jmath} + 0\hat{k}$ and putting the picture in the plane $\boxed{z=0}$

Green's thm:

$$\int_C F \cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

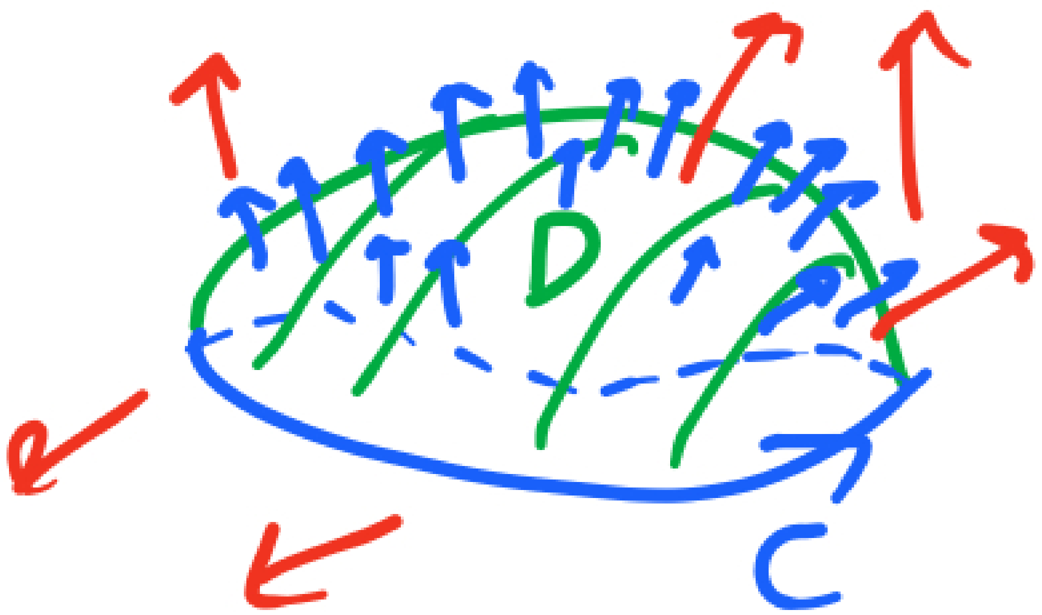
$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = (0 - 0)\,\hat{i}\;(0 - 0)\,\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

So $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = \text{curl } F \cdot \hat{k}$

And

$$\boxed{\int_C F \cdot dr = \iint_D (\text{curl } F \cdot \hat{k})\, dA}$$

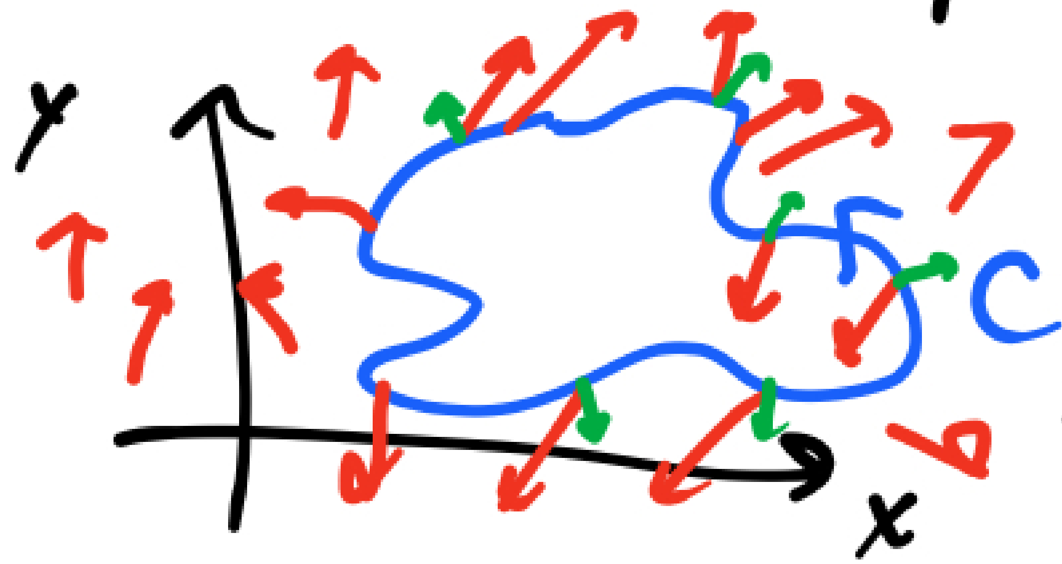
Generalization : Stokes's theorem



$$\int_C F \cdot dr = \underbrace{\iint_D (\operatorname{curl} F \cdot \vec{n})\, dA}_{\text{flux}}$$

~~Another vector form~~:

We expressed the line integral of tangential component of F along C as the double integral of the vertical component of curl F over D enclosed by C. We will look for a formula involving the normal component of F:



$C: r(t) = x(t)\hat{i} + y(t)\hat{j}$

$a \le t \le b.$

$$\vec{T}(t) = \frac{x'(t)}{\|r'(t)\|}\hat{i} + \frac{y'(t)}{\|r'(t)\|}\hat{j}$$

– unit tangent vector

$$\vec{n}(t) = \frac{y'(t)}{\|r'(t)\|}\hat{i} - \frac{x'(t)}{\|r'(t)\|}\hat{j}$$

– unit normal vector.

(e.g. $x(t) = \cos t$, $y(t) = \sin t$, then $\vec{n}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$).

$$\int_C F \cdot \vec{n}\, ds = \int_a^b (F \cdot n)(t)\, \|r'(t)\|\, dt =$$

$$= \dots = \int_a^b (Py' - Qx')\, dt =$$

$$= \int_C P\,dy - Q\,dx \overset{\text{Green's thm}}{=} \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA =$$

$$= \iint_D (\operatorname{div} F)\, dA$$

Generalization! Divergence theorem

+1 dimension