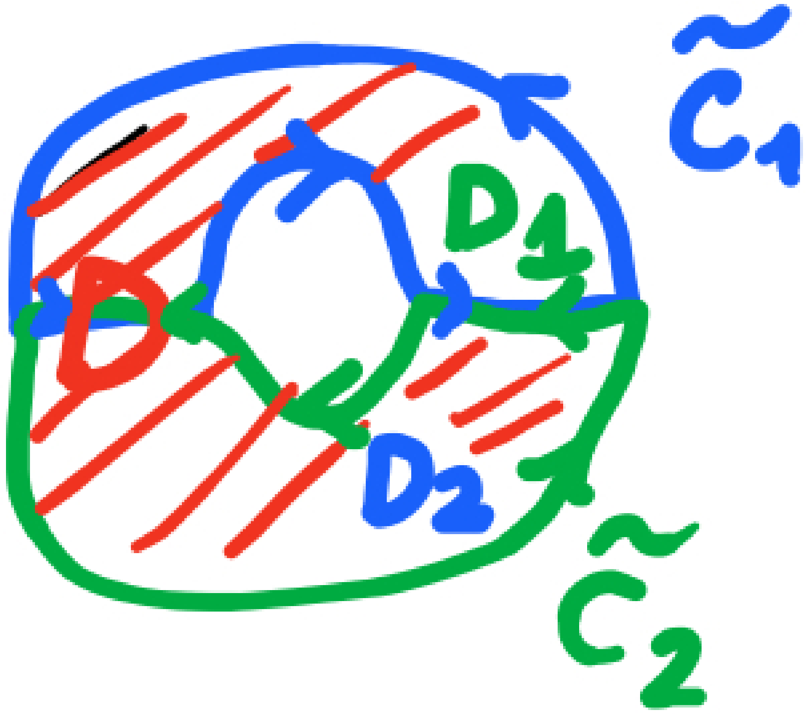
Another note on orientation:



- $$\int_{\tilde{C}_1} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

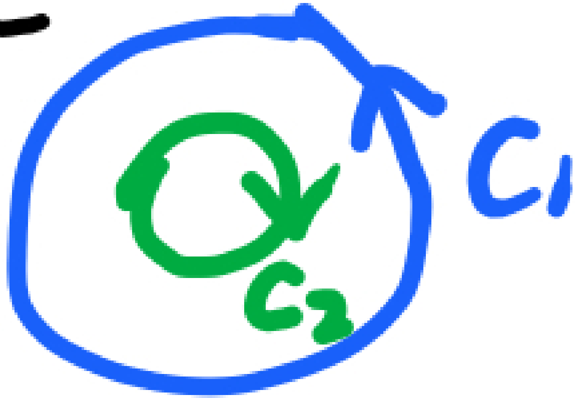
- $$\int_{\tilde{C}_2} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

D_2

Take the sum:

$$\int_C P\,dx + Q\,dy = \iint_{D=D_1\cup D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\|$$

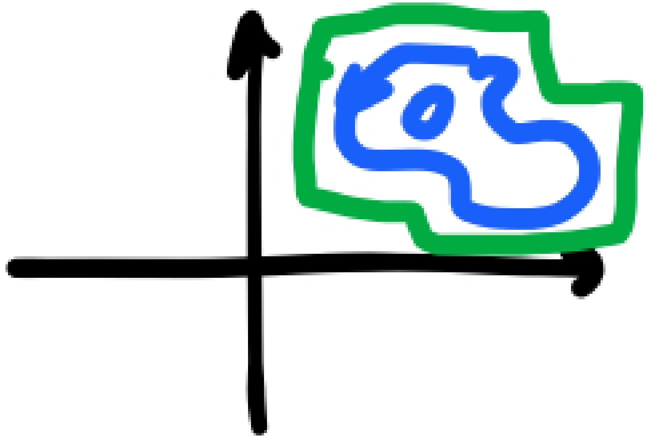


$$\int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$

Quick note: $F = -\frac{y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$

(problem from last time)

$f = \theta \in [0, 2\pi)$ – not "well-defined" at the origin



potential function

16.5. Curl and Divergence.

Important operations on vector fields

curl F — vector field

div F — function

Curl (only makes sense in 3d)

Let $F = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ be a vector field in $\mathbb{R}^3$, such that all partial derivatives of P, Q, R exist.

Then

$$\operatorname{curl} F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

How to remember?

Let "∇" be a vector with components $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

(notational)

Consider formal cross product

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} =$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}.$$

So $\boxed{\operatorname{curl} F = \nabla \times F}$ ← easy to remember

Motivation: curl helps answer the question – when F is conservative?

<u>Example</u> Find curl F for

$$F = xy\,\hat{\imath} + x^2 z\,\hat{\jmath} - y^2\,\hat{k}.$$

$$\text{curl}\,F = \nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & x^2 z & -y^2 \end{vmatrix} =$$

$$= (-2y - x^2)\,\hat{\imath} + (0)\,\hat{\jmath} + (2xz - x)\,\hat{k}.$$

<u>Theorem</u> If $f(x,y,z)$ has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \vec{0}.$$

(as a vector field)

In other words, if F is conservative, then $\text{curl}\, F = \vec{0}$.

Explanation:

$$\text{curl}(\nabla f) = \nabla \times (\nabla f) =$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix} =$$

$$= \underbrace{(f_{zy} - f_{yz})}_{=0 \text{ by Clairaut's thm}} \hat{i} + \overbrace{(f_{xz} - f_{zx})}^{=0} \hat{j} + \underbrace{(f_{yx} - f_{xy})}_{=0} \hat{k}$$

Example Show that

$$F = xy\,\hat{i} + x^2 z\,\hat{j} - y^2\,\hat{k}$$

is not conservative.

$\rightarrow$ curl $F \neq \vec{0}$ (see above).

Theorem (converse) If F is a vector field defined everywhere in $\mathbb{R}^3$ and curl $F = \vec{0}$, then F is conservative.

Example : next time.