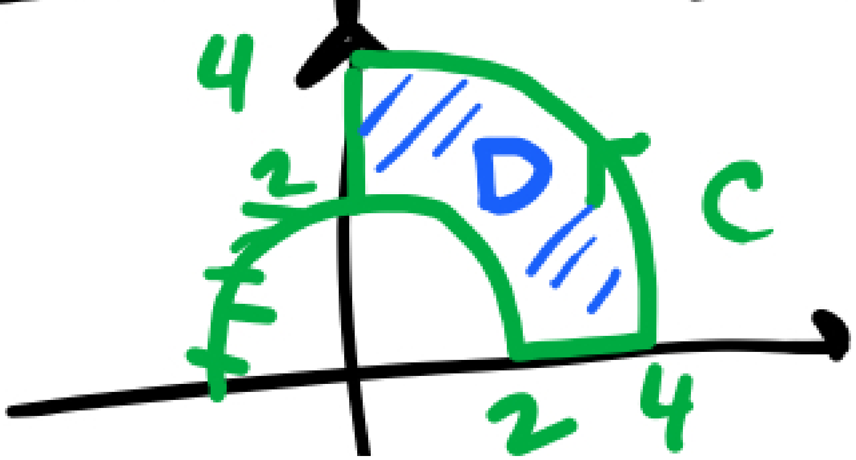
# Green's thm:


$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

**Example** Find $\int_C x^2y\,dx - y^2\,dy =$



$$= \iint_D (0 - x^2)\, dA = \int_0^{\pi/2}\int_2^4 -(r\cos\theta)^2\, r\, dr\, d\theta =$$

$$= -\int_0^{\pi/2} \cos^2\theta\, d\theta \cdot \int_2^4 r^3\, dr =$$

$$= -\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta \cdot \left.\frac{r^4}{4}\right|_2^4 =$$

$$= -\left.\frac{\theta + \frac{1}{2}\sin 2\theta}{2}\right|_0^{\pi/2} \cdot \frac{256-16}{4} =$$

$$= -\frac{\pi}{4}\cdot 60 = -15\pi.$$

# Note on orientation


positive orientation of boundary of $D$ = region is on the left along direction of $C$

boundary of $D = C =$
$= C_1 \sqcup C_2$

- Green's theorem extends to domains like that.

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

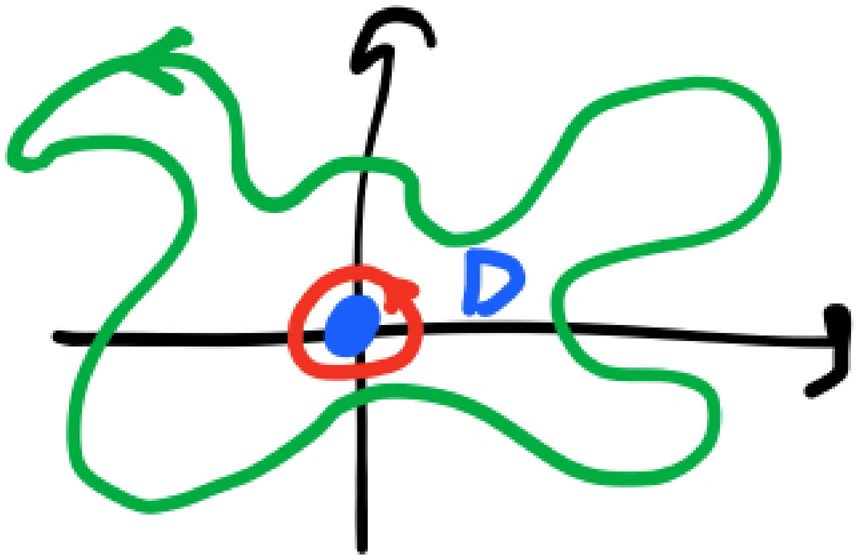
pos. oriented in the new sense

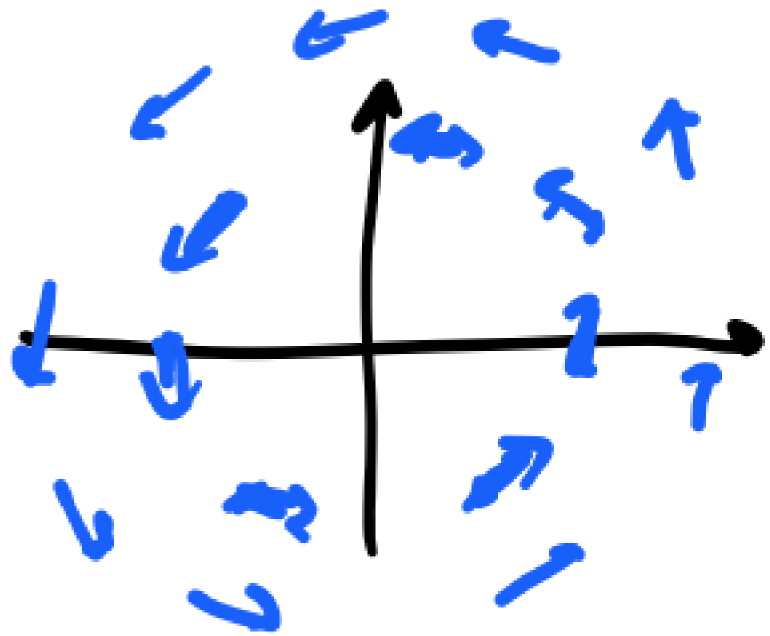
$$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$

orientation given by the picture above

**Example** (tricky) For $F = -\frac{y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$

show $\int_C F \cdot dr = 2\pi$ for **every** positively oriented ~~clos~~ simple closed curve $C$ that encloses the origin.



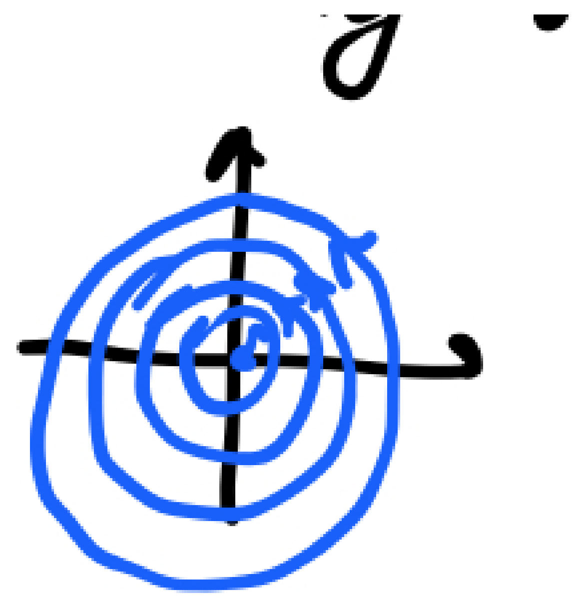



Let's check for circles centered at the origin:

$x(t) = a\cos t$

$y(t) = a\sin t$ $\qquad a > 0$

$0 \le t \le 2\pi$



$$\int_C F\cdot dr = \int \underline{F(r(t))}\cdot r'(t)\,dt =$$

$$= \int_0^{2\pi}\left(-\frac{\cancel{a}\sin t}{a^{\cancel{2}}}\hat{i} + \frac{\cancel{a}\cos t}{a^{\cancel{2}}}\hat{j}\right)\cdot\left(-a\sin t\,\hat{i} + a\cos t\,\hat{j}\right)dt =$$

$$= \int_0^{2\pi}(\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} dt = \boxed{2\pi}$$

- Show for arbitrary C

find small circle $C'$ inside $C$, want to show

$$\int_C F \cdot dr = \int_{C'} F \cdot dr$$

$= 2\pi$ (we know)

or in other words,

$\int_C F \cdot dr - \int_{C'} F \cdot dr = 0$

$$\rightarrow \int_{C \sqcup -C'} F \cdot dr = 0.$$

By extended Green's thm:

$$\int_{C \sqcup -C'} F \cdot dr = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA =$$

D: domain between $C$ and $C'$

$$= \iint_D \left[ \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} \right] dA =$$

Exercise

$$= \iint_D 0 \, dA = 0. \quad \text{finish.}$$

<u>Remark</u> F is not defined at the origin.