16.4. Green's theorem

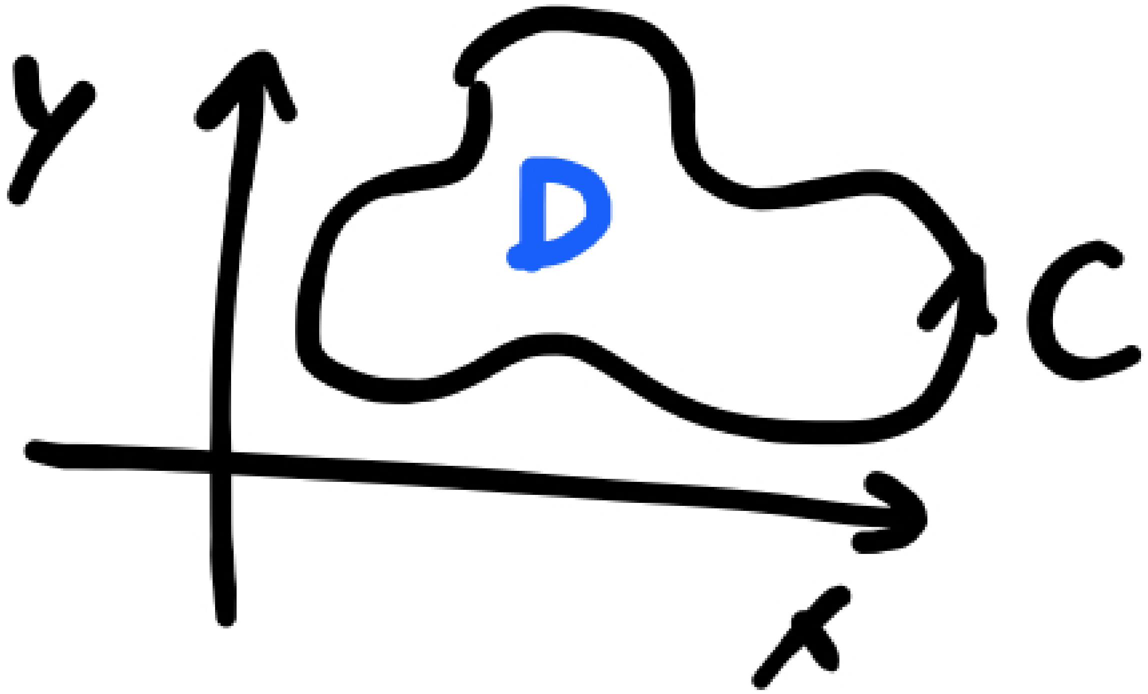
F is (continuous) conservative vector field in $\mathbb{R}^2$ $\longleftrightarrow$ $\int_C F \cdot dr = 0$ for all closed loops C.

Then if F is not conservative, there should be some closed loop s.t.

$$\int_C F \cdot dr \neq 0.$$

If further F has continuous partials of its component functions, then $\dfrac{\partial Q}{\partial x} \neq \dfrac{\partial P}{\partial y}$

(at least somewhere)



Question: Is there a relationship between line integral around C and a double integral over D bounded by C?

Green's theorem Let C be positively oriented (= counterclockwise (convention)) piecewise-smooth simple closed curve C (simple = no self-intersections) and D be the region bounded by it.

Then

$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

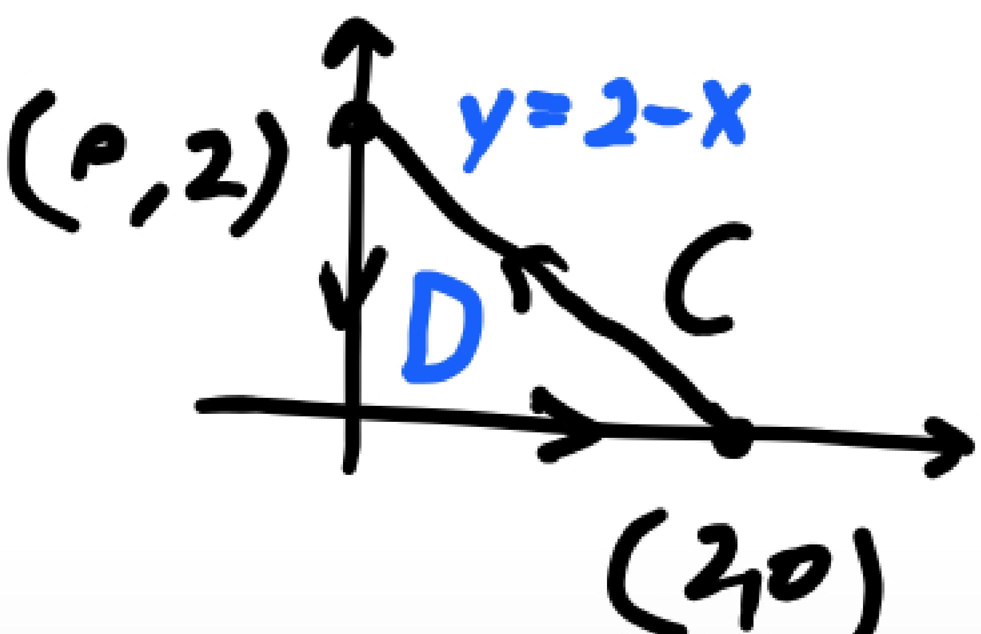
$\int_C F \cdot dr$ (for $F = P\hat{\imath} + Q\hat{\jmath}$) $=\int_C P\,dx + Q\,dy$

assuming $\frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}$ are continuous.

Note if F is conservative, Green's thm says $0 = 0$.

Remark Green's thm "measures" the failure of the fundamental thm for line integrals for non-conservative fields

Examples Find $\int_C x^2dx + xy^2dy$ for



$$\int_C x^2dx + xy^2dy \overset{\text{Green's}}{=} \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA =$$

$$= \iint_D (y^2 - 0)\, dA \overset{\text{Fubini}}{=} \int_0^2 \int_0^{2-x} y^2 \, dy\, dx =$$

$$= \int_0^2 y^3/3 \Big|_0^{2-x} dx = \int_0^2 \frac{(2-x)^3}{3} dx$$

$$= -\frac{(2-x)^4}{3 \cdot 4} \Big|_0^2 = \frac{2^4}{3 \cdot 4} = 4/3 .$$

<u>Example</u> Find $\oint_C (2y - x^x)\, dx + (x + \sqrt{y^{10} + y})\, dy.$

notation for pos. oriented s.c.c.

$C: x^2 + y^2 = 4.$

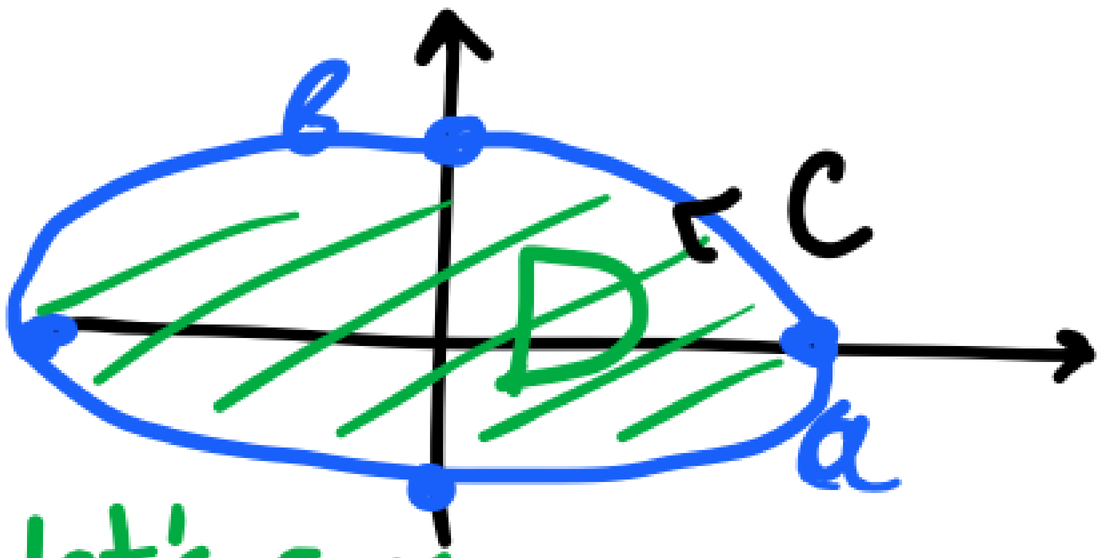
$$\oint_C \cdots\cdots = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA =$$

$$= \iint_D (1 - 2) \, dA = \iint_D -1 \, dA =$$

$$= -1 \iint_D 1 \, dA = -1 \cdot \pi 2^2 =$$

$$= \boxed{-4\pi}$$

<u>Application</u> Find the area ~~of~~ enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



$$\text{Area} = \iint_D \underset{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}{\underset{\shortparallel}{1}}\, dA =$$

let's say

$P = y$

$Q = 2x$

$$= \int_C P\,dx + Q\,dy =$$

$$= \int_C y\,dx + 2x\,dy =$$

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$$

$0 \le t \le 2\pi$

$$= \int_0^{2\pi} b\sin t \cdot (-a\sin t)\,dt$$

$$+ \; 2a\cos t \; b\cos t \; dt =$$

$$= \int_0^{2\pi} 2\underline{ab}\cos^2 t - \underline{ab}\sin^2 t\,dt$$

$$= ab \int_0^{2\pi} 2\cos^2 t - \sin^2 t \, dt =$$

Exercise

$$ab \int_0^{2\pi} 2 \cdot \frac{1+\cos 2t}{2} - \frac{1-\cos 2t}{2} \, dt =$$

$$= ab \int_0^{2\pi} 1 + \cos 2t - \frac{1}{2} + \frac{\cos 2t}{2} \, dt =$$

$$= ab \int_0^{2\pi} \frac{1}{2} + \frac{3}{2} \cos 2t \, dt =$$

$$= ab \ \cancel{\int} \ \frac{t}{2} + \frac{3}{4} \sin 2t \Big|_0^{2\pi} = \boxed{\pi ab}$$