Fundamental theorem for line integrals

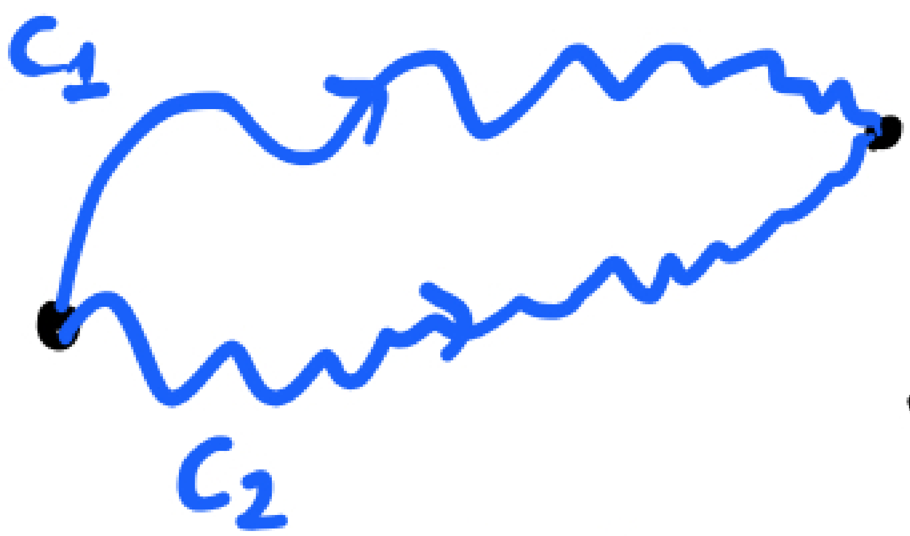
Let C be a (piecewise) smooth plane (or space) curve given by vector function $\vec{r}(t)$, $a \le t \le b$

Let $f(x,y)$ (or $f(x,y,z)$) be a differentiable function with continuous gradient vector field $\nabla f(x,y)$ ($\nabla f(x,y,z)$)

Then

$$\int_C \nabla f \cdot dr = f(\vec{r}(b)) - f(\vec{r}(a)).$$

__Upshot__ : Line integrals of conservative vector fields are independent of path.

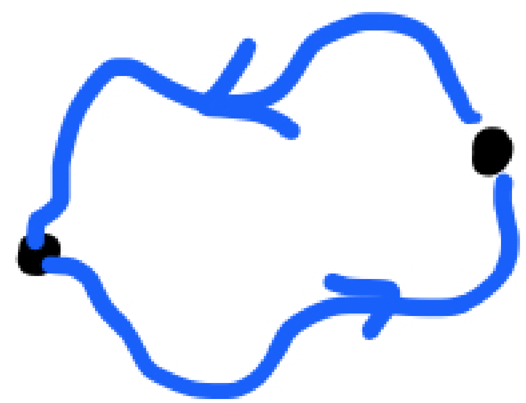


$$\int_{C_1} \overset{\text{conservative}}{F} \cdot dr = \int_{C_2} \overset{\text{conservati}}{F} \cdot dr$$

Let's focus on $\mathbb{R}^2$

__Theorem__ Let F be a continuous vector field in $\mathbb{R}^2$, such that $\int_C F \cdot dr$ is independent of path. Then there is a function f, such that $\nabla f = F$.

Observation



"$\int_C F \cdot dr$ is independent of path in $\mathbb{R}^2$" equivalent to

→ "$\int_C F \cdot dr = 0$ for all <u>closed</u> paths"

Remark Recall $\int_C F \cdot dr = 2\pi$ for $F = \langle y, -x \rangle$ and C - unit circle (oriented clockwise)

Theorem Let $F = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ be a vector field in $\mathbb{R}^2$, such that P and Q have continuous ~~second order~~ partial derivatives. Then F is conservative if and only if

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}\ !$$

"Explanation": if $F = \nabla f$, then

$\Longrightarrow$

$$F = \underset{\overset{\shortparallel}{P}}{f_x}\,\hat{\imath} + \underset{\overset{\shortparallel}{Q}}{f_y}\,\hat{\jmath}$$

$$\frac{\partial P}{\partial y} = (f_x)_y \qquad \frac{\partial Q}{\partial x} = (f_y)_x$$

By Clairaut's theorem, $f_{xy} = f_{yx}$.

Hence $\underbrace{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$

Warning the converse ($\Leftarrow$) of the theorem does not hold if we replace $\mathbb{R}^2$ with domains with holes

Example Is $F = (x+2y)\hat{\imath} + (x-y)\hat{\jmath}$

conservative?

$$\frac{\partial P}{\partial y} = 2 \neq \frac{\partial Q}{\partial x} = 1$$

$\Rightarrow$ not conservative.

<u>Question</u> If F is conservative (e.g. we checked $\partial P/\partial y = \partial Q/\partial x$ and the domain doesn't have holes), how to find $\underline{\underline{f}}$?

<u>Motivation</u>: we want to apply FTL to compute line integrals.

<u>Examples</u>

$$F = (3+2xy)\hat{\imath} + (x^2-3y^2)\hat{\jmath}$$

- $\partial P/\partial y = 2x \;=\; \partial Q/\partial x = 2x$ ✓

  F is <u>conservative</u>.

- looking for f such that

  $f_x = 3+2xy \qquad f_y = x^2-3y^2$

I. Integrate f_x w.r.t x :
(as a function of x only)

$$f = \underline{\underline{3x + x^2 y}} + \underbrace{C(y)}_{\text{function of } y}$$

II. Differentiate f w.r.t. y :

$$f_y = 0 + x^2 + \underline{C'(y)}$$

$$= x^2 - \underline{3y^2}$$

$$\Rightarrow C'(y) = -3y^2$$

$$\Rightarrow C(y) = -y^3 + D.$$

Overall $f = 3x + x^2 y - y^3 + \underline{\underline{D}}.$