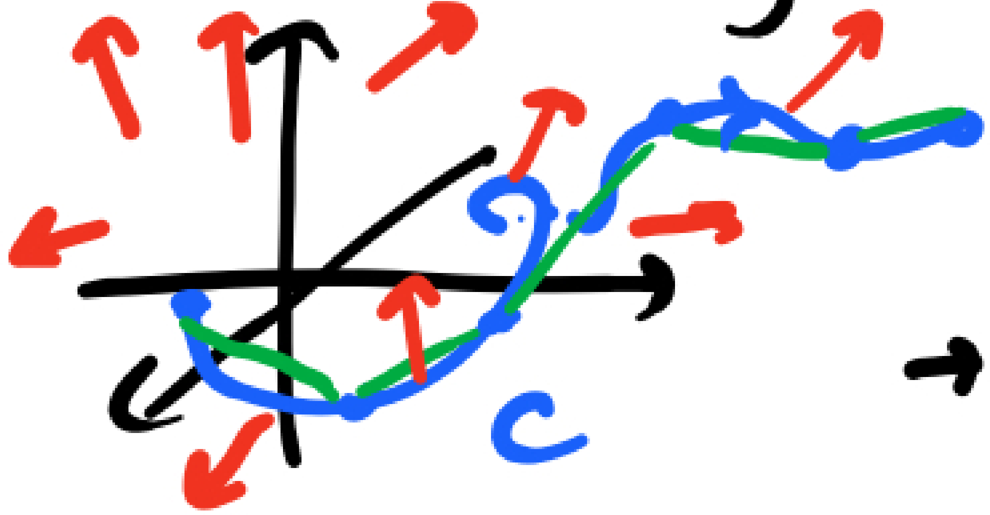
- Line integrals in 3D



Similarly, we can define:

$\rightarrow \int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$

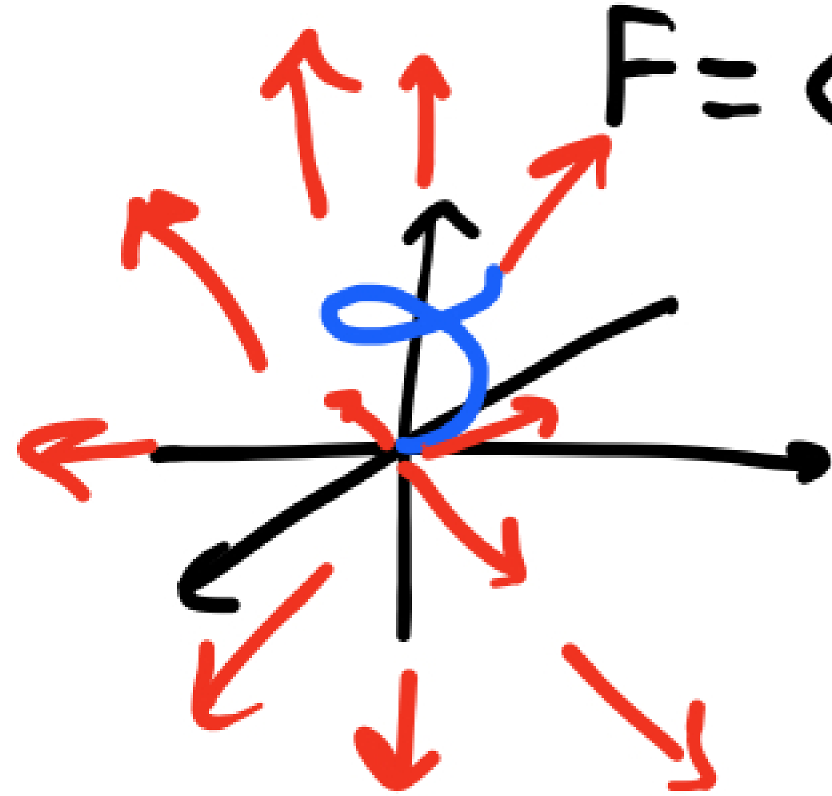
$\rightarrow \int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\,dt$

$\rightarrow \int_C F \cdot dr = \int_C P\,dx + Q\,dy + R\,dz$

if $F = P(x,y,z)\hat{\imath} + Q\hat{\jmath} + R\hat{k}$

Example $\int_C F \cdot dr$, where $F = \langle x, y, z \rangle$

and $C =$



$x = \cos t$
$y = \sin t$
$z = t$
$0 \le t \le 2\pi$

$$F(r(t)) = \langle \cos t, \sin t, t \rangle$$
$$r'(t) = \langle -\sin t, \cos t, 1 \rangle$$
$$\int_C F \cdot dr = \int_0^{2\pi} \langle \cos t, \sin t, t \rangle \cdot \langle -\sin t, \cos t, 1 \rangle \, dt$$
$$= \int_0^{2\pi} (\cancel{-\cos t \sin t} + \cancel{\sin t \cos t} + t \cdot 1) \, dt =$$
$$= \int_0^{2\pi} t \, dt = t^2/2 \Big|_0^{2\pi} = 4\pi^2/2 = 2\pi^2.$$

16.3. The Fundamental Theorem for Line Integrals

first in the series of fund. thms of vector calculus

- **Recall**: $$\boxed{\int_a^b F'(x)\,dx = F(b) - F(a)}$$

  - Fundamental theorem of Calculus

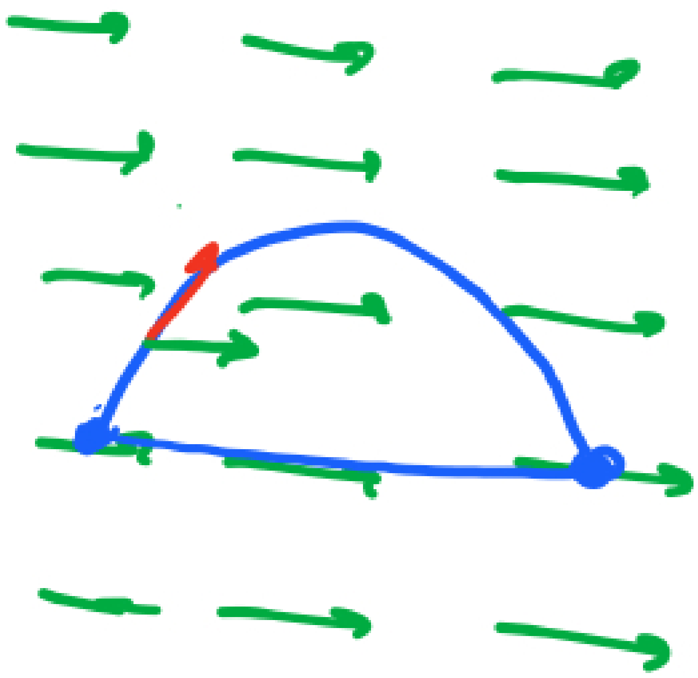
In multivariable calculus, we replace F' with ∇F:

Theorem Let C a piecewise smooth curve given by vector function $\vec{r}(t)$, $a \le t \le b$

Let $f(x,y)$ be a differentiable function with continuous gradient vector field $\nabla f(x,y)$. Then

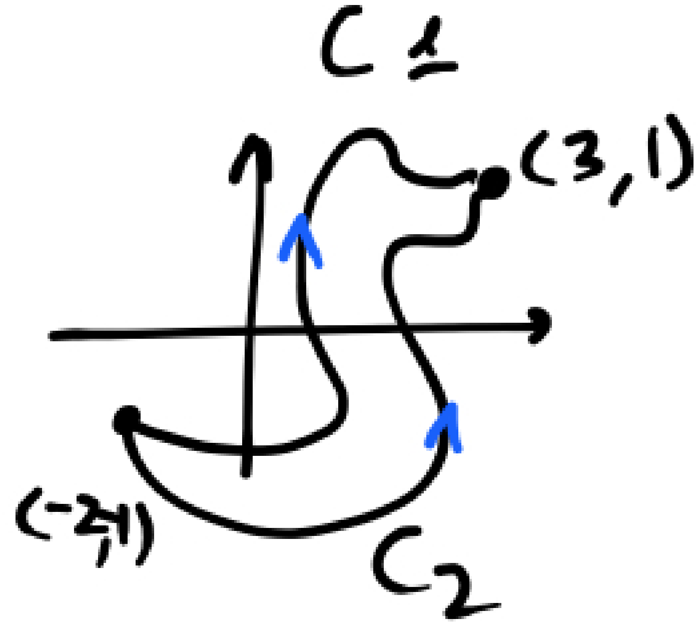
$$\boxed{\int_C \nabla f \cdot dr = f(\vec{r}(b)) - f(\vec{r}(a))}$$

Remark the value of the integral depends on the endpoints of curve C only.



Example

- $\int_{C_1} \nabla f \cdot dr$
- $\int_{C_2} \nabla f \cdot dr$



$f(x,y) = x^2+y^2$

$$\int_{C_1} \nabla f \cdot dr = \int_{C_2} \nabla f \cdot dr =$$

$$= f(3,1) - f(-2,-1) =$$

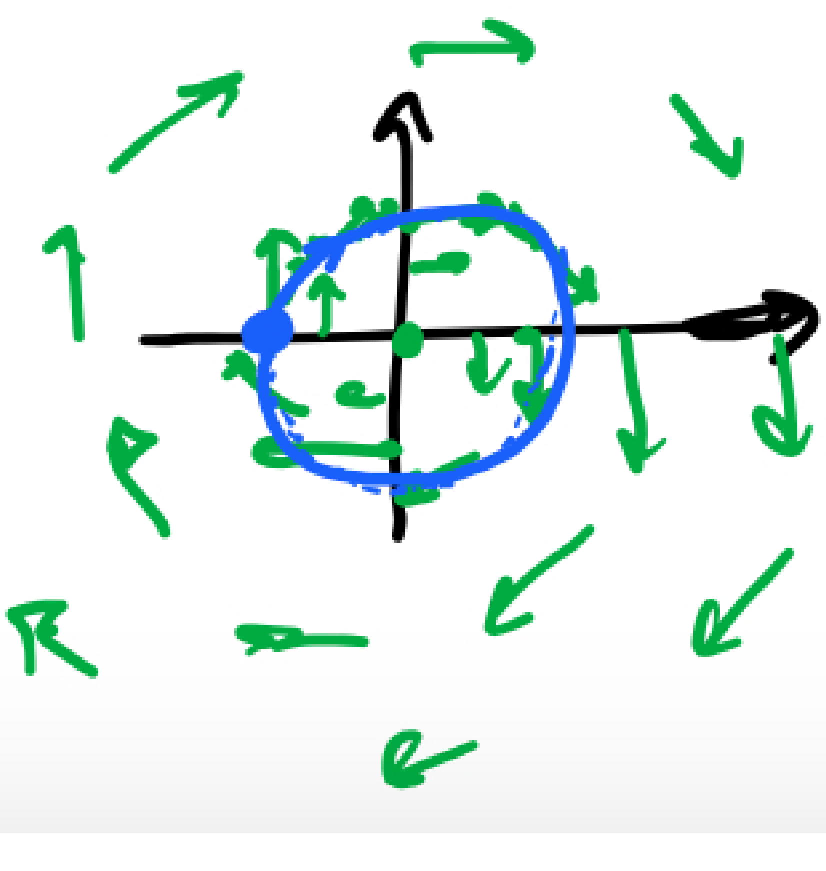
$$= 3^2+1^2 - \left((-2)^2+(-1)^2\right) =$$

$$= 10 - 5 = \boxed{5}$$

Def We say F (vector field) is **conservative** if it is a gradient vector field of some function.

Example (non consenative vector field)

$$F = \langle y, -x \rangle .$$



F is not gradient vector field of any function.

(1) Start from a point and follow the field. Since gradient points in the direction of max. increase after a full circle we will get a bigger value at the same point - contradiction.

(2) similar but more rigorous

$$\int_C F \cdot dr = 2\pi \quad \text{(did it before)}$$

C unit circle

BUT by the Fundamental thm,

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr =$$

$$= f(-1,0) - f(-1,0) = 0$$

contradiction.

(3) Suppose there is $f(x,y)$ s.t.

$$\nabla f = f_x \hat{\imath} + f_y \hat{\jmath} = y\,\hat{\imath} - x\,\hat{\jmath}.$$

Then $(f_x)_y = 1.$

$(f_y)_x = -1.$

But $f_{xy} = f_{yx}$ so $1 = -1$

Contradiction

Next time : how to determine whether F is conservative in general; if F is conservative, how to find f ?!