Last time:

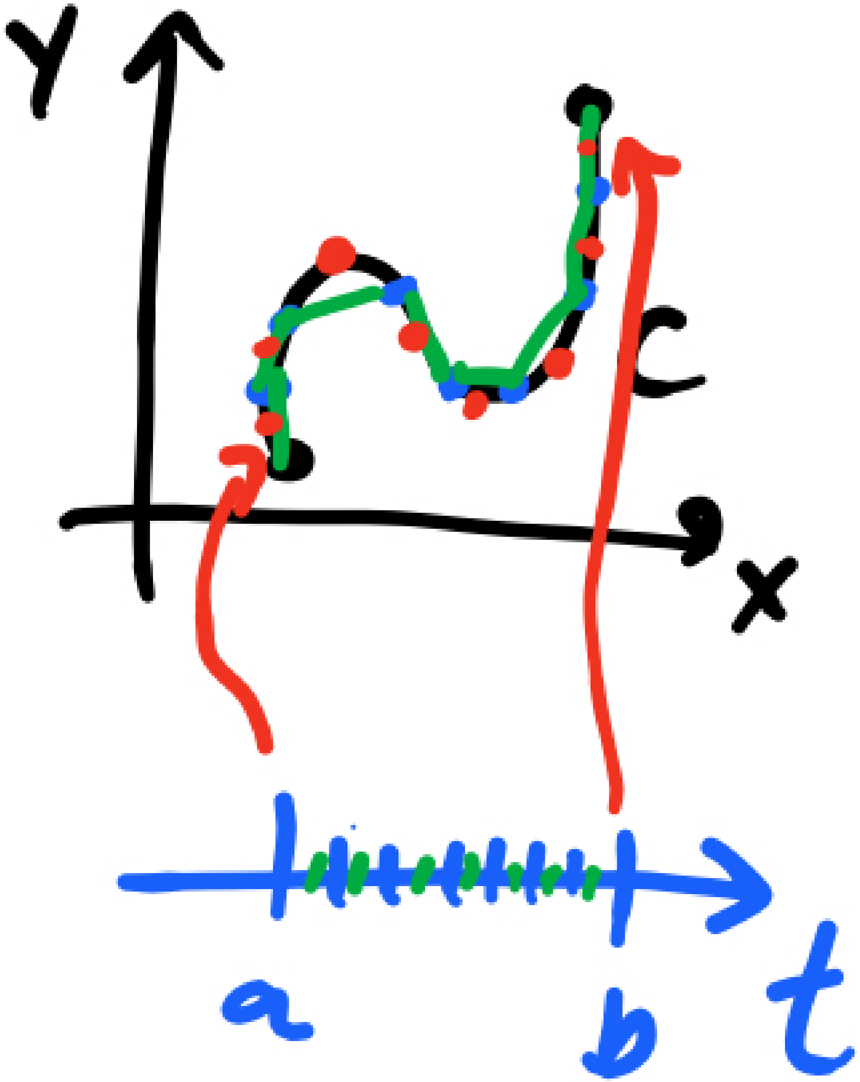
- ★ Line integrals of a vector field

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

Today:

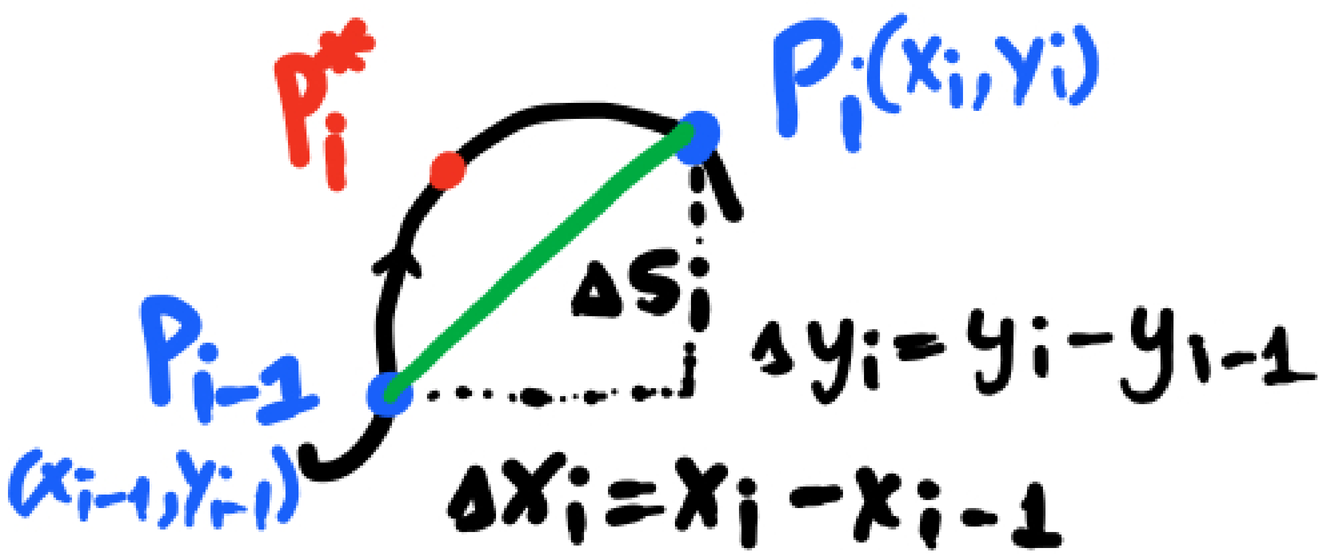
- line integrals with respect to x,
  - - - - to y.

  (as a specific case of ★)



Suppose we are given

- $f(x,y)$
- C is given by a parametrization $x(t), y(t), a \le t \le b$



Def

$$\int_C f(x,y)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*,y_i^*)\cdot\Delta x_i\right]$$

$$\int_C f(x,y)\,dy = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*,y_i^*)\cdot\Delta y_i\right]$$

- If $x = x(t)$, $y = y(t)$

then $dx = x'(t)\,dt$

$dy = y'(t)\,dt$

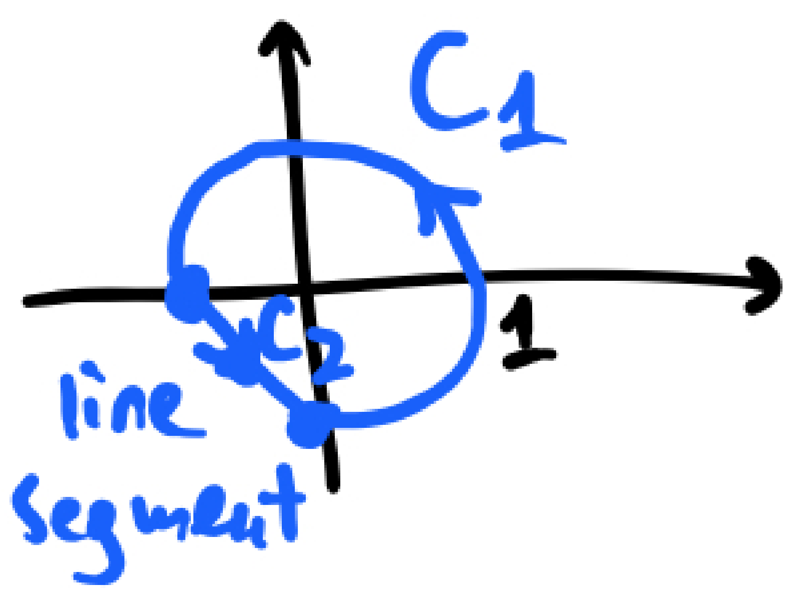
$$\boxed{\begin{aligned}\int_C f(x,y)\,dx &= \int_a^b f(x(t),y(t))\,x'(t)\,dt\\ \int_C f(x,y)\,dy &= \int_a^b f(x(t),y(t))\,y'(t)\,dt\end{aligned}}$$

we can also talk about their combinations:

$$\int_C f(x,y)\,dx + g(x,y)\,dy$$

<u>Example</u> $\int_C x\,dy + y\,dx$ for

C :



C_1 : $x(t) = \cos t \quad y(t) = \sin t \qquad -\frac{\pi}{2} \le t \le \pi$

C_2 : $x(t') = t'$

$y(t') = -t' - 1 \qquad -1 \le t' \le 0$

$$\int_C x\,dy + y\,dx = \int_{C_1} x\,dy + y\,dx + \int_{C_2} x\,dy + y\,dx =$$

$$= \int_{-\pi/2}^{\pi} \cos t \cdot \cos t\,dt + \sin t(-\sin t)\,dt$$

$$+ \int_{-1}^{0} t(-1)\,dt + (-t-1)\,dt =$$

$$= \int_{-\pi/2}^{\pi} \cos^2 t - \sin^2 t\,dt + \int_{-1}^{0} -2t - 1\,dt$$

$$= \int_{-\pi/2}^{\pi} \cos 2t\,dt + -t^2 - t \Big|_{-1}^{0} =$$

$$= \frac{1}{2}\sin 2t \Big|_{-\pi/2}^{\pi} + (0-0) =$$

$$= \frac{1}{2}(\sin 2\pi - \sin(-\pi)) = 0.$$

Relationship with line integral of a vector field.

$\rightarrow \int_C F \cdot dr$

Let's say

$$\mathbf{F} = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$$

and also

$$\mathbf{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$$

$$a \leq t \leq b$$

$F(r(t)) =$

$P(x(t), y(t))\hat{\imath} + Q(x(t), y(t))\hat{\jmath}$

$r'(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath}$

$$\int_C F \cdot dr \overset{def}{=} \int_a^b F(r(t)) \cdot r'(t)\, dt =$$

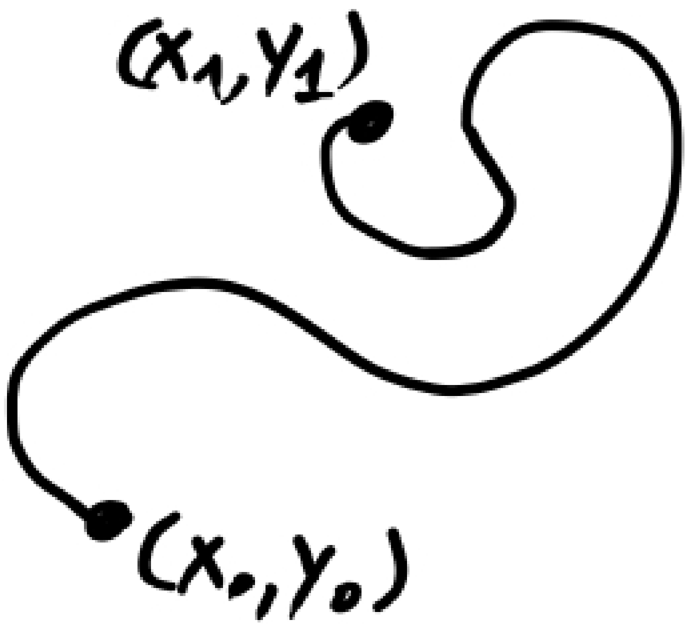
$$= \int_a^b P(x(t), y(t)) \cdot x'(t)\, dt + \int_a^b Q(x(t), y(t))\, y'(t)\, dt =$$

$$= \boxed{\int_C P(x,y)\, dx + Q(x,y)\, dy}$$

Upshot: $\int_C F \cdot dr = \int_C P\,dx + Q\,dy$

for $F = P\hat{\imath} + Q\hat{\jmath}$.

Next time: • fundamental theorem for line integrals.



$$\int_C \underline{\underline{\nabla f}} \cdot dr = f(x_1, y_1) - f(x_0, y_0)$$

• examples of line integrals in 3D.