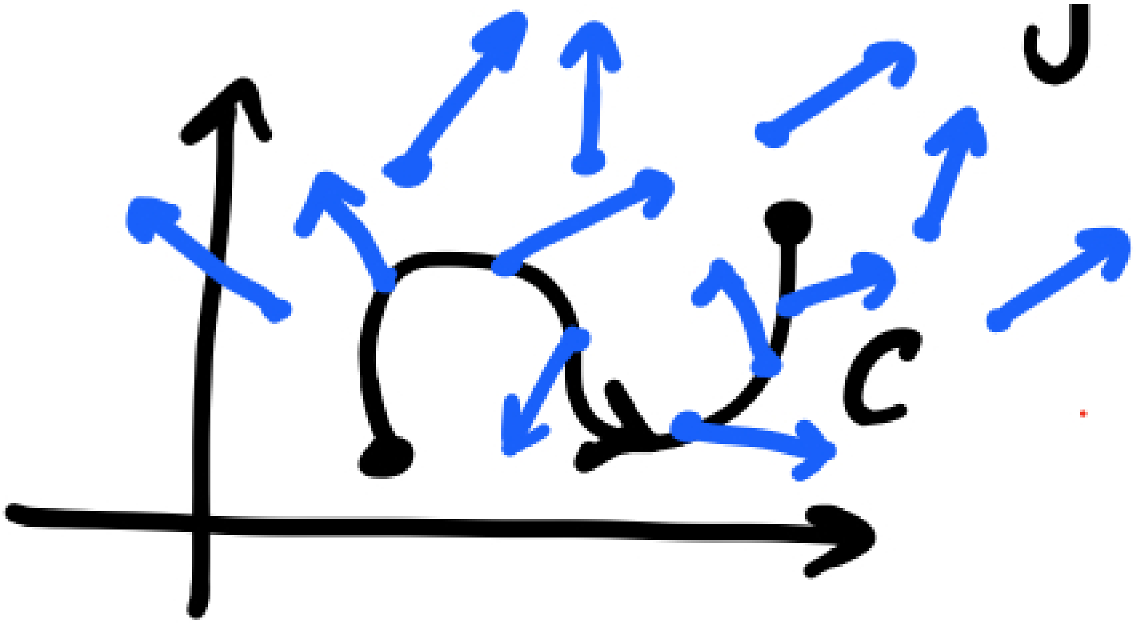
Line Integrals

Last time: line integral of $f(x,y)$ with respect to arclength along a curve C

$$\int_C f(x,y)\,ds$$

Today: line integral of a vector field along C



• Suppose we are given a vector field

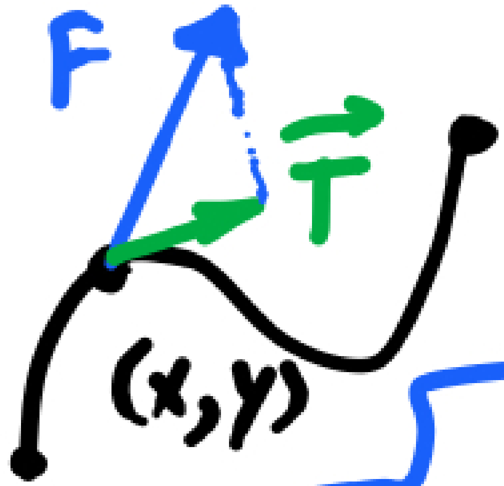
$$F = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$$

and a curve C given by

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}, \quad a \le t \le b.$$

Interpretation of the line integral: work done by the force field F to move a particle along C.

field along C.



$\vec{T}(x,y)$ - unit tangent vector to C

$$\boxed{W = \int_C \underbrace{F(x,y)\cdot\vec{T}(x,y)}_{\text{magnitude of the projection of F onto T}}\, ds}$$

$$\vec{T}(\cancel{x,y})(t) = \underbrace{\vec{r}\,'(t)}_{\text{some tangent vector}} \Big/ \|\vec{r}\,'(t)\|$$

$$\text{"}ds = \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}_{\text{"} \|\vec{r}\,'(t)\|}\, dt\text{"}$$

$$W = \int_a^b F(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}} \cancel{\|\vec{r}\,'(t)\|}\, dt$$

$$= \boxed{\int_a^b F(\vec{r}(t)) \bullet r'(t)\, dt} = \int_C F \bullet dr$$

notation

line integral of F along C.

<u>Remark</u> the curve C needs to be oriented to make sense of line integral. Usually, C is given through a parametrization that has the induced orientation.

Example $F(x,y) = y\hat{\imath} - x\hat{\jmath}$

and C is unit cirle centered at zero, counterclockwise orientation.

$$\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath}$$

$$0 \leq t \leq 2\pi$$

$$\int_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t)\,dt$$

$$= \int_0^{2\pi} (\sin t\,\hat{\imath} - \cos t\,\hat{\jmath}) \cdot (-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath})\,dt$$

$$= \int_0^{2\pi} \underbrace{-\sin^2 t - \cos^2 t}_{-1}\,dt =$$

$$\int_0^{2\pi} -1\,dt = -2\pi.$$