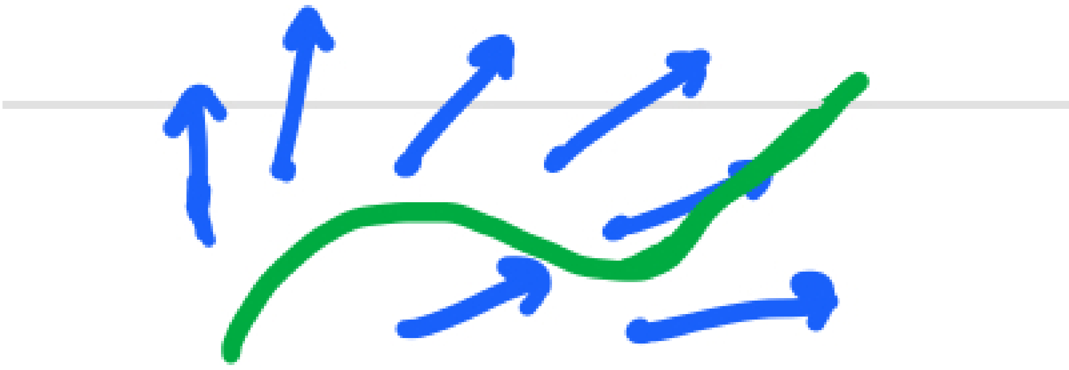
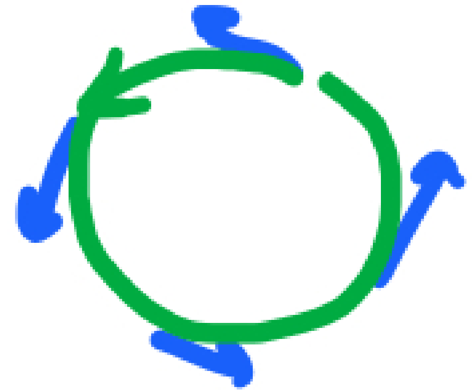
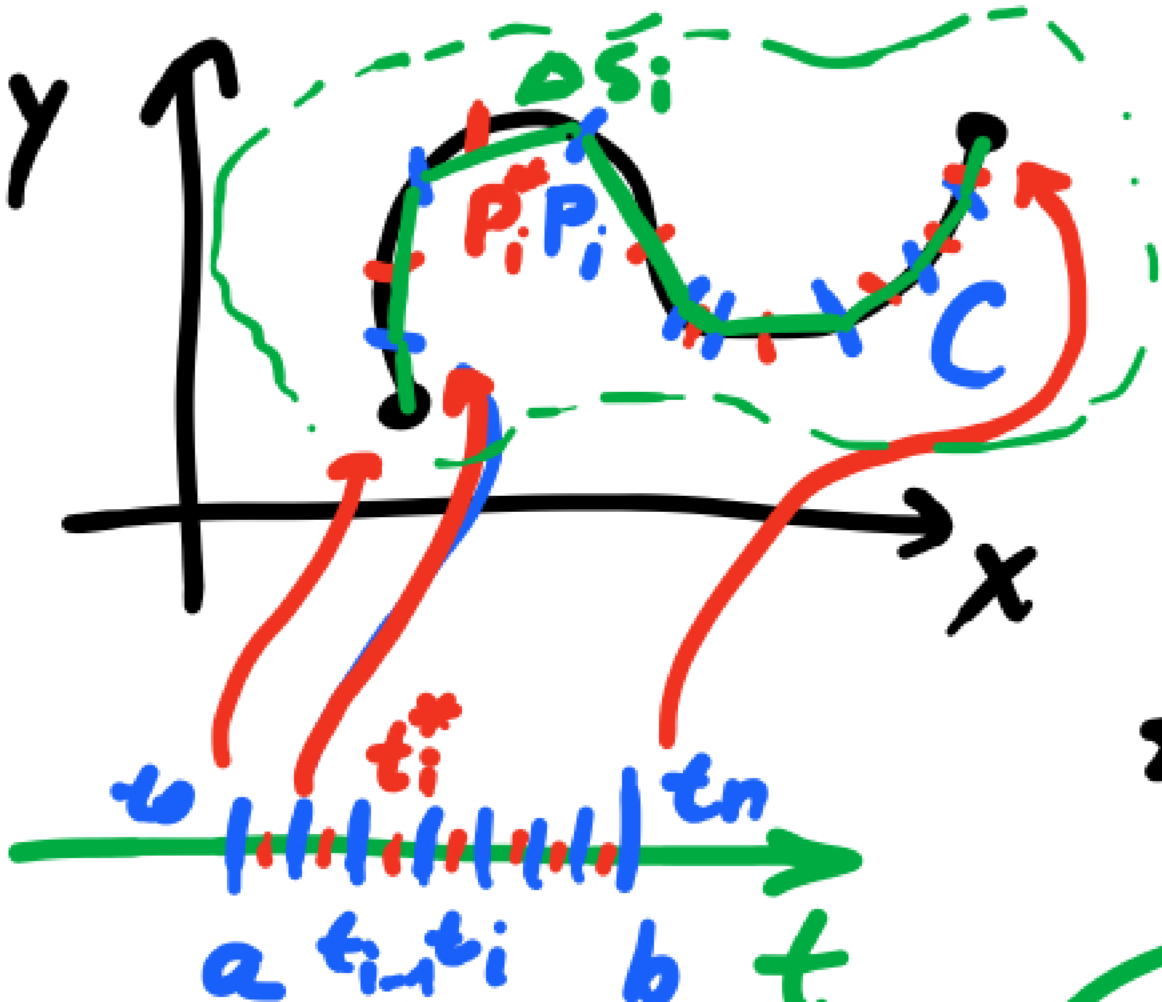




16.2. Line Integrals :



- ★ line integral of $f(x,y)$ along C w.r.t. arclength
- ★ line integral of a vector field F along C

special cases: w.r.t. x and w.r.t y

Suppose plane curve C is given by parametric equation : $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

$a \le t \le b$

- Divide $[a,b]$ into n subintervals $[t_{i-1}, t_i]$
  pick $t_i^* \in [t_{i-1}, t_i]$

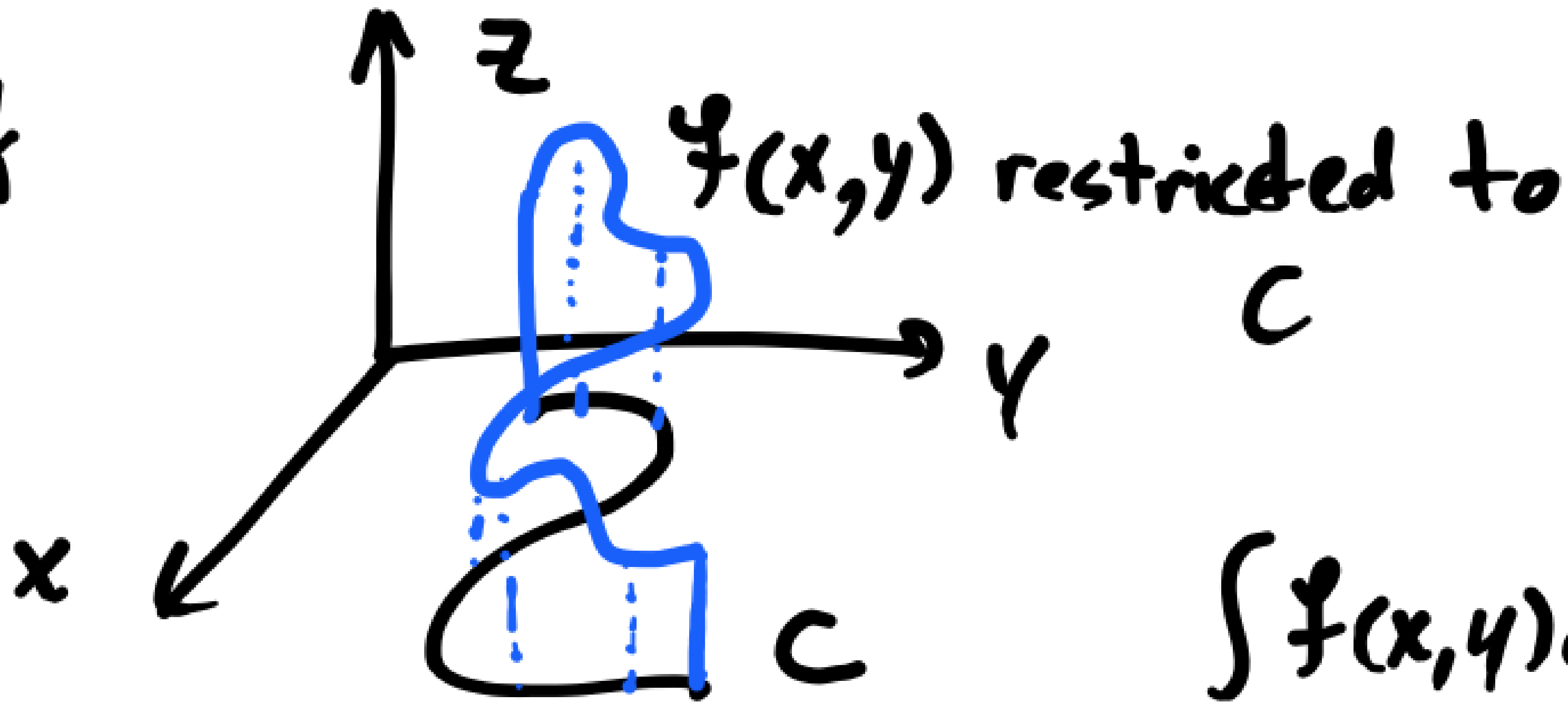
- Consider pts $P_i(x_i, y_i)$, $P_i^*(x_i^*, y_i^*) \in C$ that correspond to t_i's and t_i^*'s, and connect P_i's with subarcs of length ΔS_i.

Consider Riemann sum: $\displaystyle\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta S_i$

Def The line integral of f along C is w.r.t. arclength

$$\boxed{\int_C f(x,y)\,ds} = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i\right).$$

Remark



$\int_C f(x,y)\,ds$ equals to the area of the "curtain"

Remark If $f=1$, $\int_C f\,ds = \text{length}(C)$.

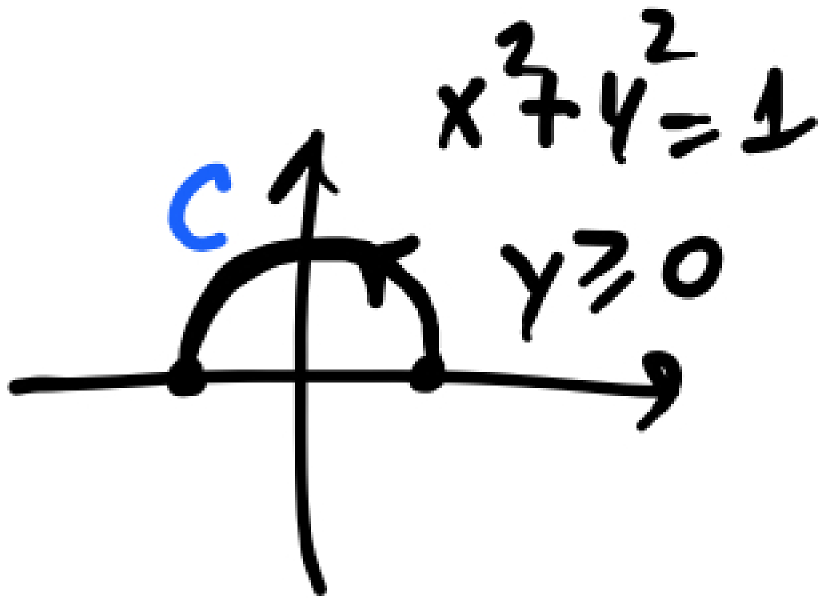
on the other hand,

$$\text{length of } C = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Therefore

$$\boxed{\int_C f(x,y)\,\underbrace{ds} = \int_a^b f(x(t),y(t))\,\underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}}$$

Example $\int_C (x^2+y^2)\,ds$, where



· parmetrize C

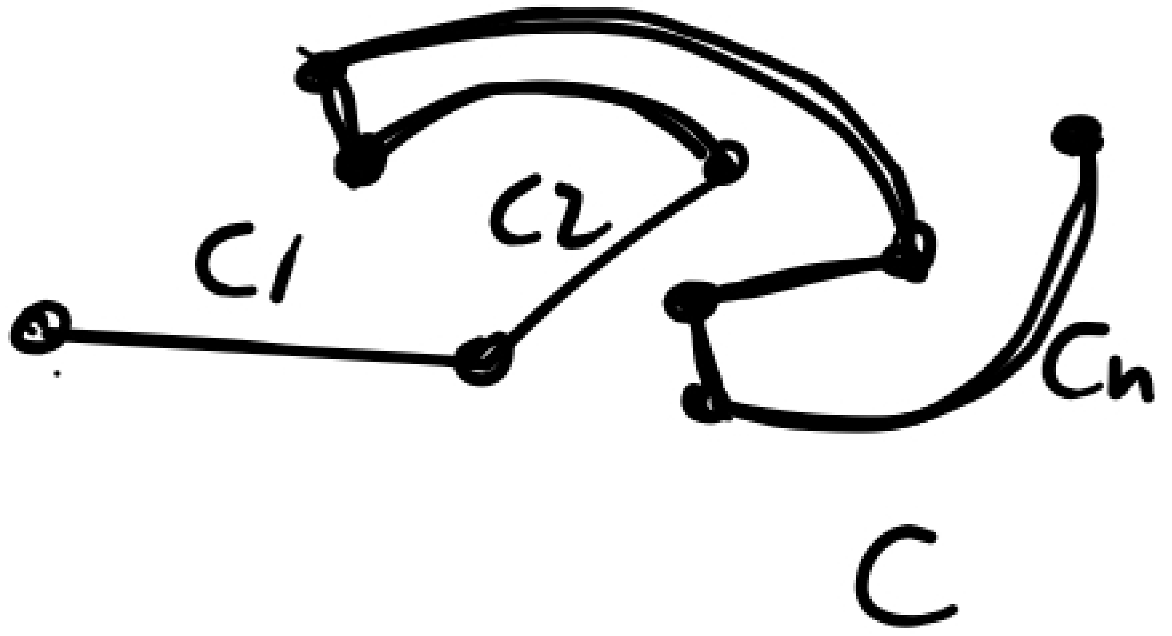
$x = \cos t \qquad y = \sin t \qquad 0 \le t \le \pi$

· compute

$$\int_C x^2+y^2\,ds = \int_0^\pi \overbrace{\cos^2 t+\sin^2 t}^{1}\,\sqrt{(-\sin t)^2+(\cos t)^2}\,dt =$$

$$= \int_0^\pi 1 \cdot 1\,dt = \boxed{\pi}$$

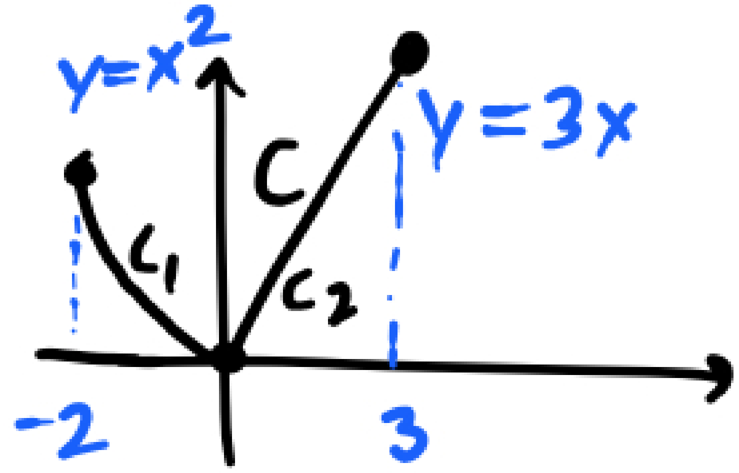
•



If C is piecewise-nice, then

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \dots + \int_{C_n} f(x,y)\,ds$$

Example $\displaystyle\int_C 2y+3x \, ds$



· parametrize C_1 and C_2

C_1 : $\quad x=t \quad y=t^2 \quad , -2\le t\le 0$

C_2 : $\quad x=t' \quad y=3t' \quad , 0\le t'\le 3$

$$\int_C 2y+3x\,ds = \int_{C_1} 2y+3x\,ds + \int_{C_2} 2y+3x\,ds =$$

$$= \int_{-2}^{0}(2t^2+3t)\sqrt{1^2+(2t)^2}\,dt +$$

$$\int_0^3 6t+3t\sqrt{1^2+3^2}\,dt =$$

= Exercise.