Jacobians

- 1d

$$\int_a^b f(x)\,dx \overset{x=g(u)}{=} \int_c^d f(g(u))\,g'(u)\,du$$

$dx = g'(u)\,du$

$a = g(c)$

$b = g(d)$

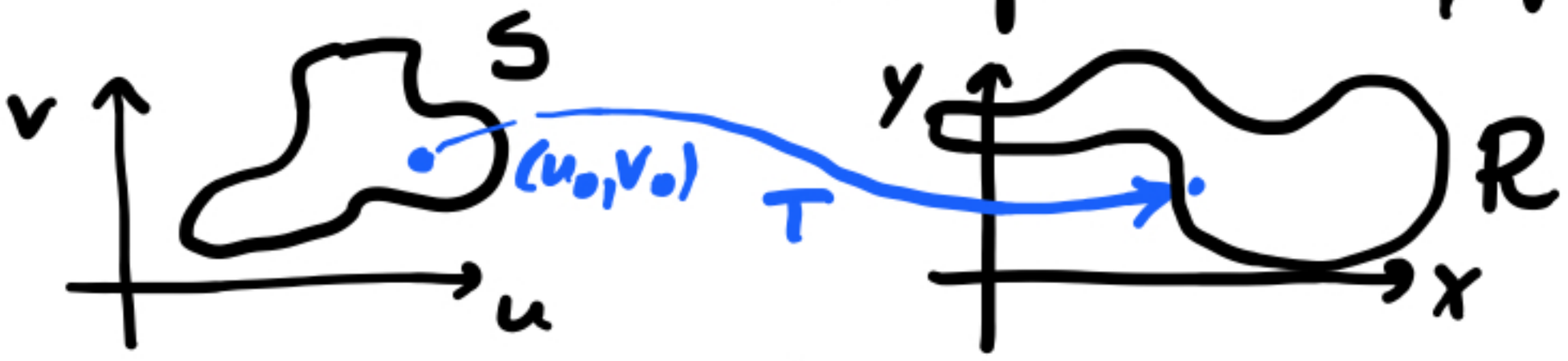
- 2d

$$\iint_R f(x,y)\,dA \overset{\substack{x=r\cos\theta \\ y=r\sin\theta}}{=} \iint_S f(r\cos\theta,\ r\sin\theta)\,r\,dr\,d\theta$$

Reminder

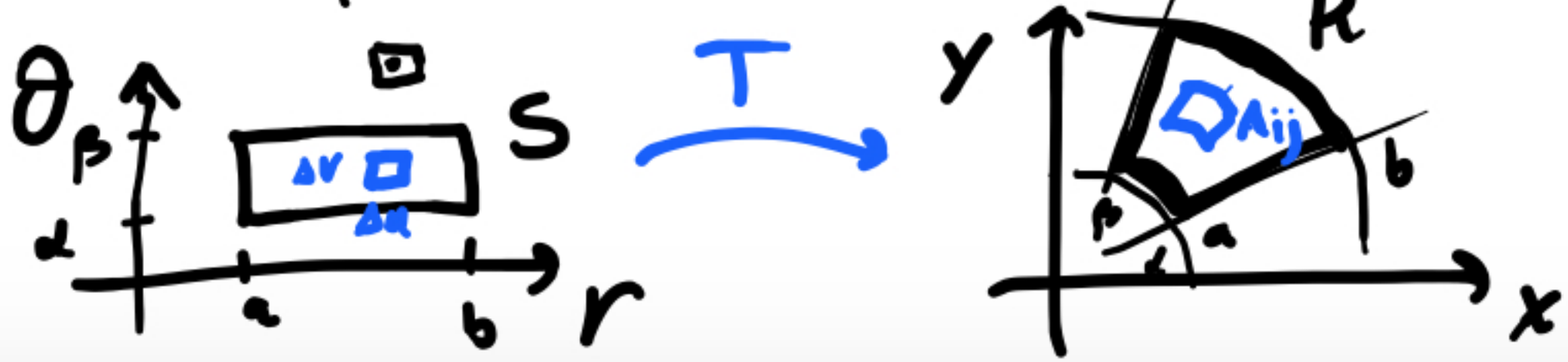
$$T \begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

Can think about change of variables as of transformation from uv-plane to xy-plane



T maps (transforms) S into R

E.g. $T \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$



What happens to the integral?

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \underbrace{\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\Delta A_{ij}}_{\text{Riemann sum}} =$$

$$= \lim \sum\sum f(g(u_i,v_j),h(u_i,v_j))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v$$

factor responsible for the change in area (multiplicative)

This factor is called <u>Jacobian</u>

Jacobian formula

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| =$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Example

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} =$$

$$= r\cos^2\theta - (-r\sin^2\theta) =$$

$$= r(\cos^2\theta + \sin^2\theta) = \boxed{r}$$

Triple integrals

$$x = g(u,v,w)$$
$$y = h(u,v,w)$$
$$z = k(u,v,w)$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Exercise Find Jacobian for spherical change of variables