Example $\displaystyle\int_1^2 \ln x\,dx \underset{\substack{x=e^u \\ dx=e^u du \\ 1=e^0 \\ 2=e^{\ln 2}}}{=\!=} \int_0^{\ln 2} u e^u\,du =$

$\overset{\text{by parts}}{=} (u-1)e^u\Big|_0^{\ln 2} = 2\ln 2 - 1$

Double integral: polar coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

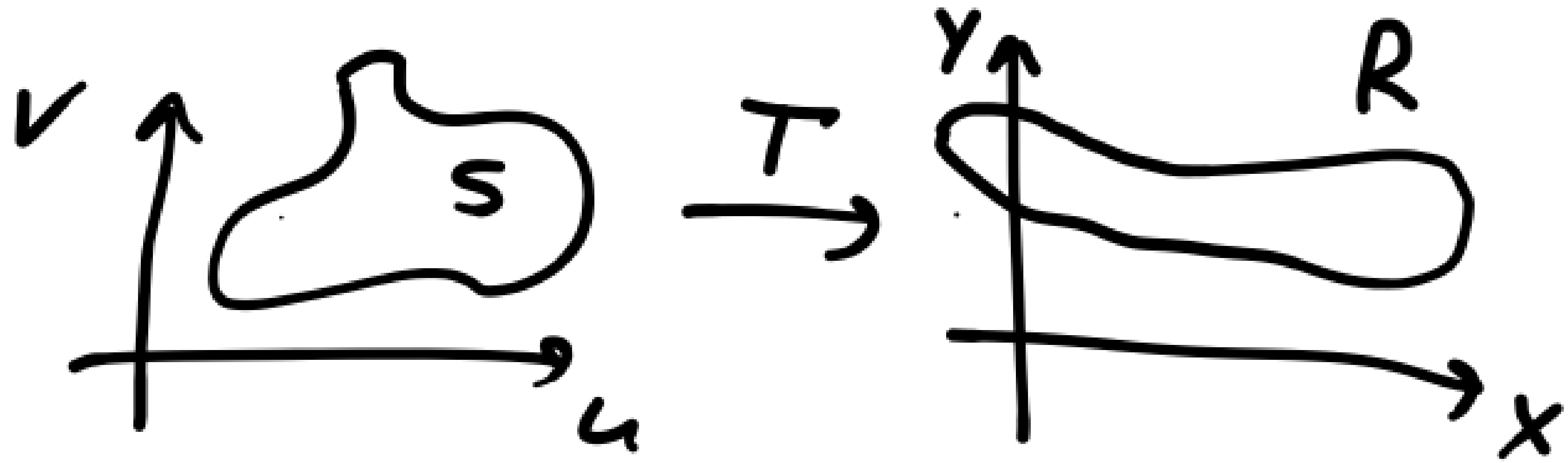
S-region in $r\theta$-plane
that corresponds to R in xy-plane

e.g. if $R = \{(x,y) \mid x^2+y^2 \le 1\}$ $S = \left\{(r,\theta) \,\middle|\, \begin{array}{l} 0\le r\le 1 \\ 0\le\theta\le 2\pi \end{array}\right\}$

## General setting

Tranformation $T$ $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$

from $uv$-plane to $xy$-plane




$T$ transforms $S$ into $R$

## Example

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$T$ transforms $S = [0,1] \times [0,2\pi]$

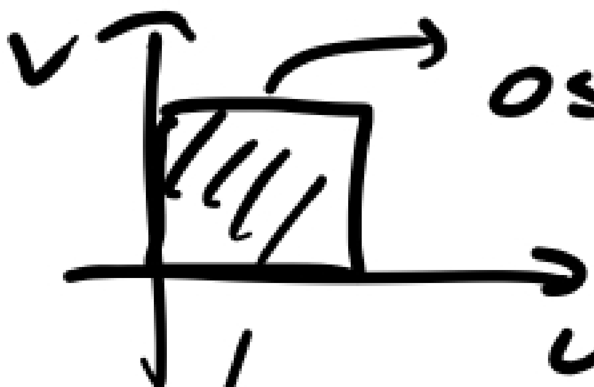
into $R = \{(x,y) \mid x^2 + y^2 \leq 1\}$

**Example** Find the image of

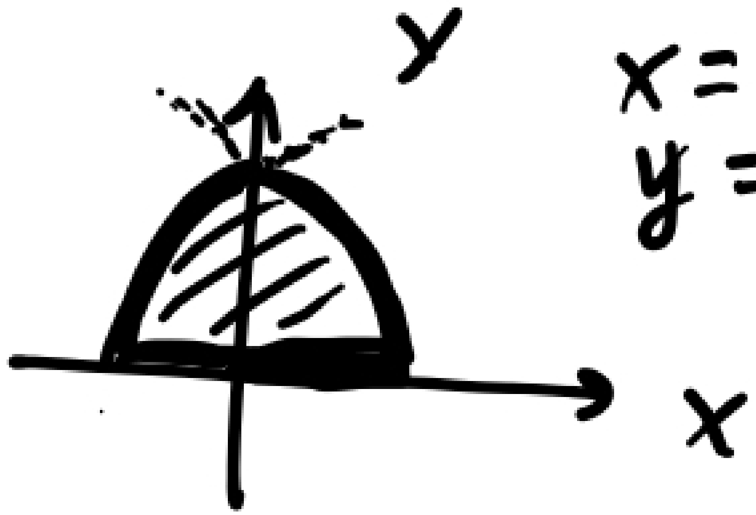
$S = \left\{ (u,v) \;\middle|\; \begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le 1 \end{array} \right\}$ under transformation

$x = u^2 - v^2$
$y = 2uv$

$v = 1$
$0 \le u \le 1$



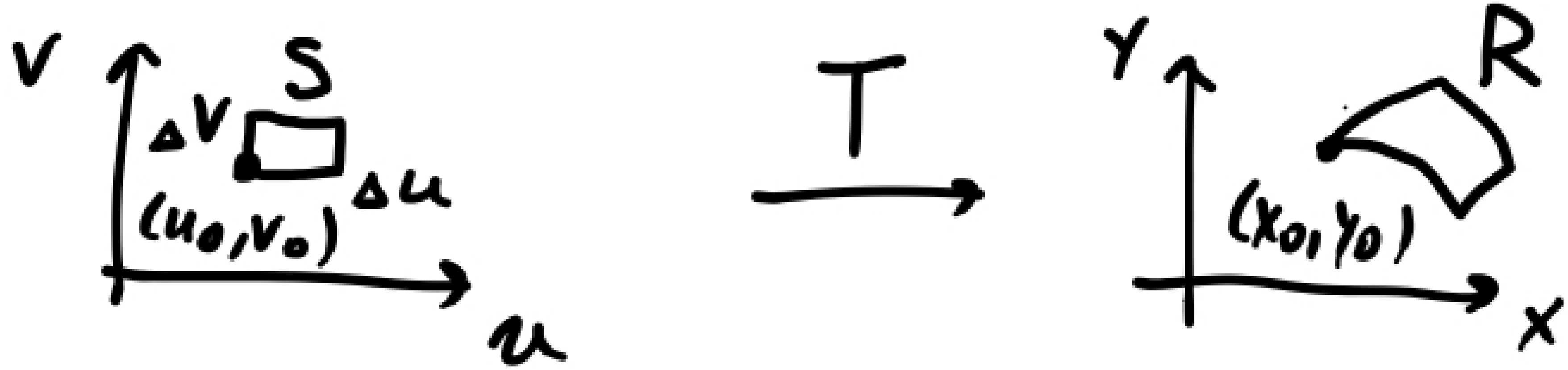

$v = 0$
$0 \le u \le 1$




$x = u^2 - 1$
$y = 2u$
$x = \frac{y^2}{4} - 1$

$x = u^2 - 0$
$y = 0$

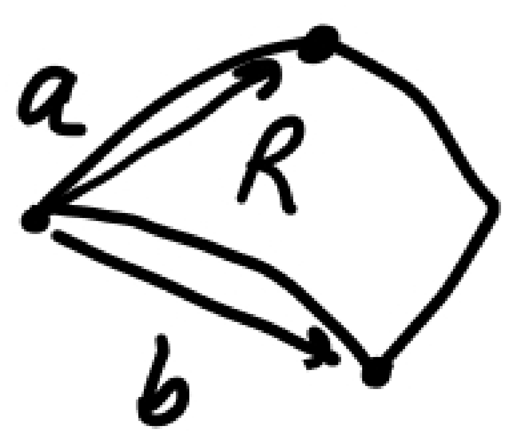
What happens to the integral?




$$\mathbf{r}(u,v) = g(u,v)\,\hat{\imath} + h(u,v)\,\hat{\jmath} + 0\,\hat{k}$$

position vector of the image of point $(u,v)$

<u>Idea</u>:
Approximate $R$ with a parallelogram




$$\vec{a} = r(u_0, v_0 + \Delta v) - r(u_0, v_0)$$

$$\vec{b} = r(u_0 + \Delta u, v_0) - r(u_0, v_0)$$

Recall, $r_u = \lim_{\Delta u \to 0} \frac{r(u_0+\Delta u, v_0) - r(u_0, v_0)}{\Delta u}$

(partial derivative)

So $\vec{a} \approx r_v \cdot \Delta v$
$\vec{b} \approx r_u \cdot \Delta u$

Approximate area of $R$:

$$\| \vec{a} \times \vec{b} \| = \| (r_u \cdot \Delta u) \times (r_v \cdot \Delta v) \| =$$

$$= \| r_u \times r_v \| \cdot \Delta u \cdot \Delta v$$

$$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}}_{\text{Jacobian of transformation } T} \hat{k}$$

## Result

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v),h(u,v)) \left| J \right| du\,dv$$

$$J = |g_u h_v - g_v h_u|$$

$g = r\cos\theta$
$h = r\sin\theta$

Exercise: find the Jacobian