Example Where is the curve

$$\begin{cases} x = \ln t \\ y = e^t \end{cases} \quad t > 0 \qquad e^x = t \qquad y = e^{(e^x)}$$

concave up?

Need $\dfrac{d^2y}{dx^2} > 0$.

chain rule

We need $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \boxed{\dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$ if $\dfrac{dx}{dt} \neq 0$

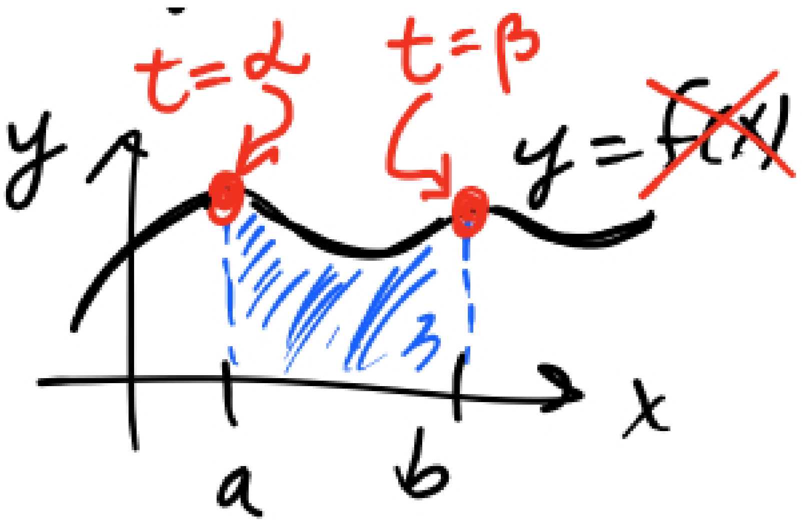
$$= \frac{\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{e^t}{1/t}\right)}{1/t} = t\,\frac{d}{dt}\left(te^t\right) =$$

$$= t\left(te^t + e^t\right) = \underbrace{t}_{>0}\underbrace{(1+t)}_{>0}\underbrace{e^t}_{>0}.$$

Answer: always concave up.

Areas

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

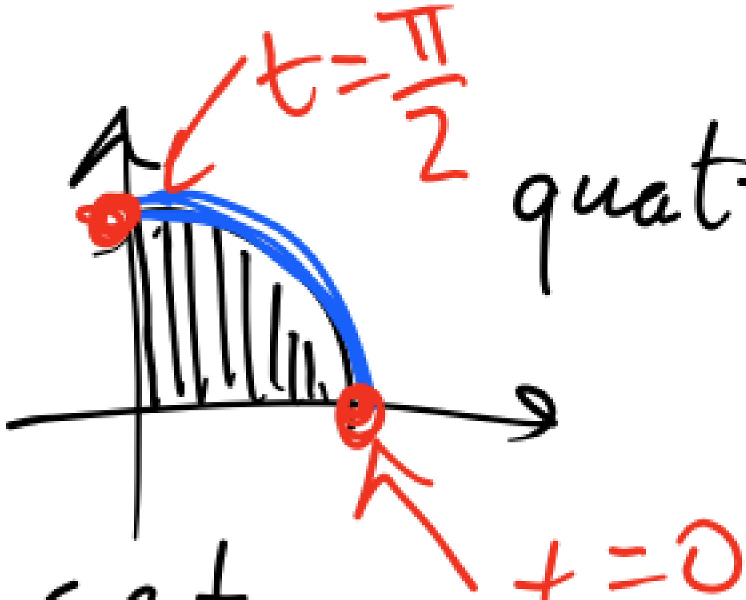


$$\text{area} = \int_a^b \cancel{f(x)}\, y \; d\cancel{x}(t)\; \frac{dx}{dt}\,dt$$

$$= \int_\alpha^\beta \left[y \, \frac{dx}{dt} \right] dt$$

$$\boxed{A = \int_\alpha^\beta \left[y(t) \, \frac{dx}{dt} \right] dt}$$

Example



quater unit disk

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

$$A = \frac{1}{4} \cdot \pi \cdot 1^2 = \frac{\pi}{4}.$$

0

$$0 \le t \le \pi/2$$

$$A = \int_{\pi/2}^{0} \sin t \cdot (-\sin t)\, dt =$$

$$= \int_0^{\pi/2} (-\sin^2 t)\, dt =$$

$$= \int_0^{\pi/2} \boxed{\sin^2 t}\, dt =$$

double angle formula

$$= \int_0^{\frac{\pi}{2}} \boxed{\frac{1-\cos 2t}{2}}\, dt =$$

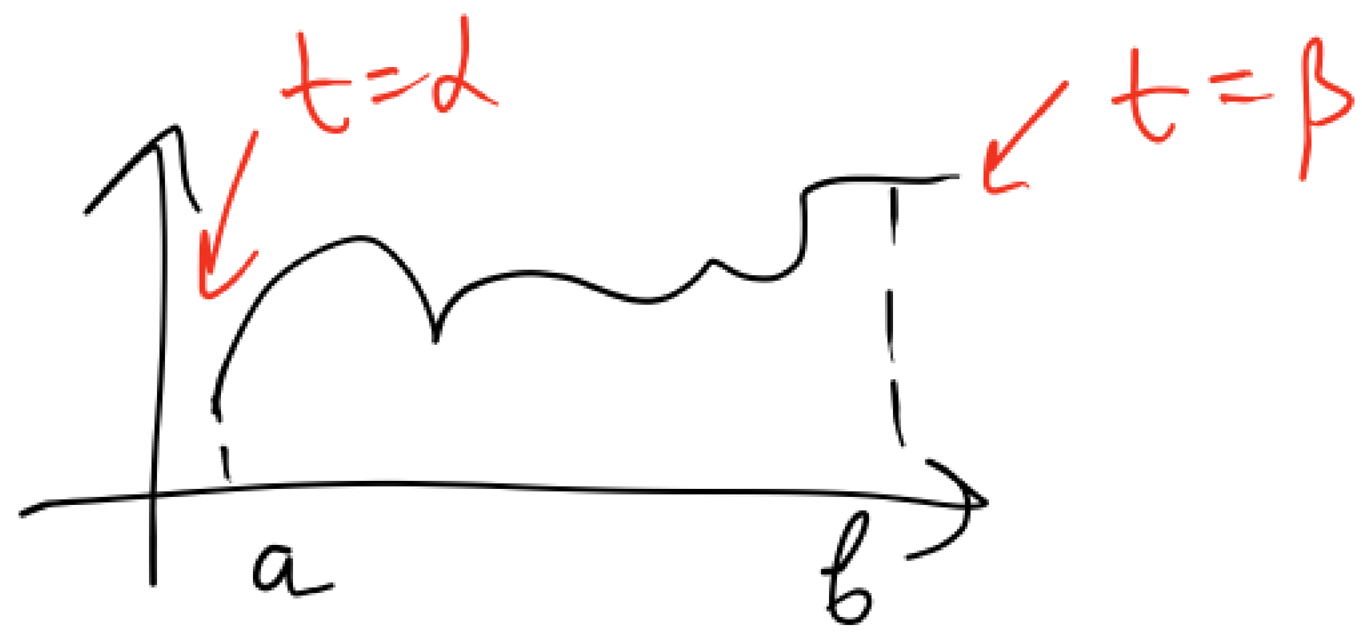
check

$$= \frac{t}{2} - \frac{\sin 2t}{4} \Bigg|_0^{\pi/2} = \boxed{\frac{\pi}{4}}$$

Arc length

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

$$\alpha \le t \le \beta$$



if $y = f(x)$, then arc length is

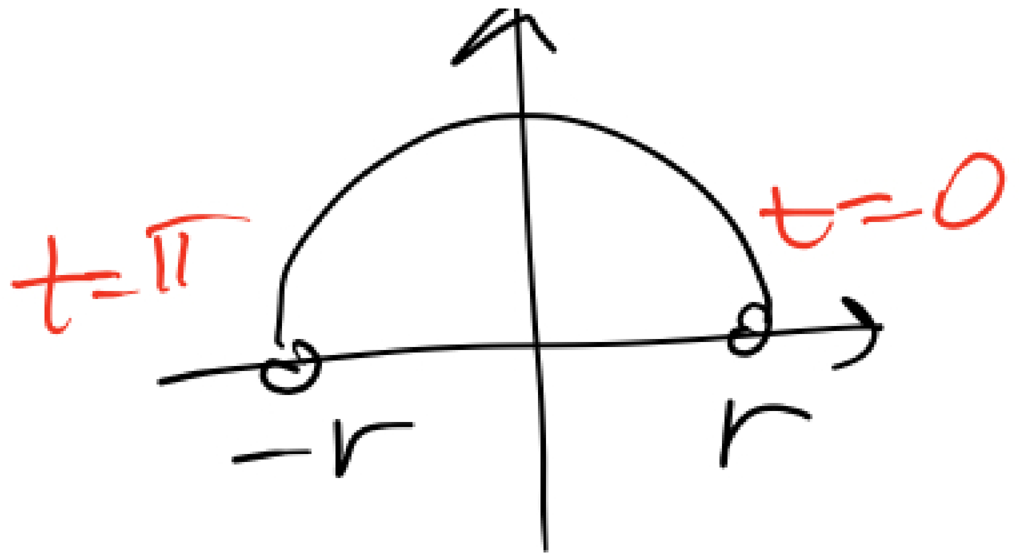
$$s = \int_{\cancel{a}\,\alpha}^{\cancel{b}\,\beta} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, d\cancel{x}\ \frac{dx}{dt}\,dt =$$

$$\left(\frac{dy/dt}{dx/dt}\right)$$

$$= \boxed{\int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}\ .$$

Example

① $\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$ $\quad 0 \le t \le \pi$

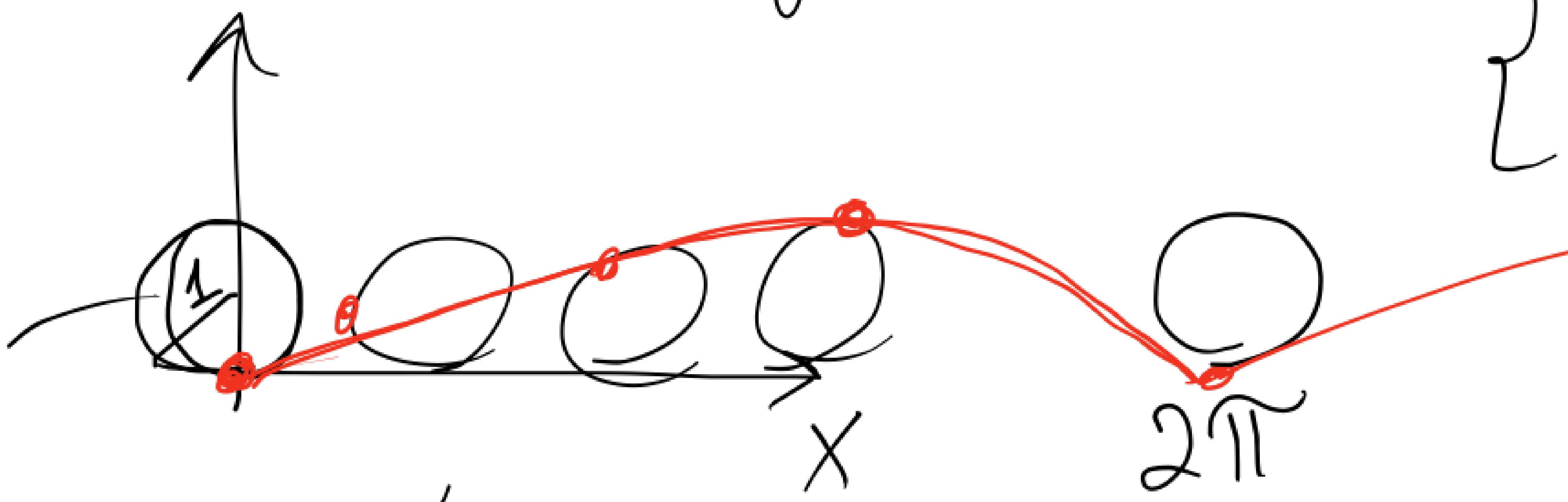


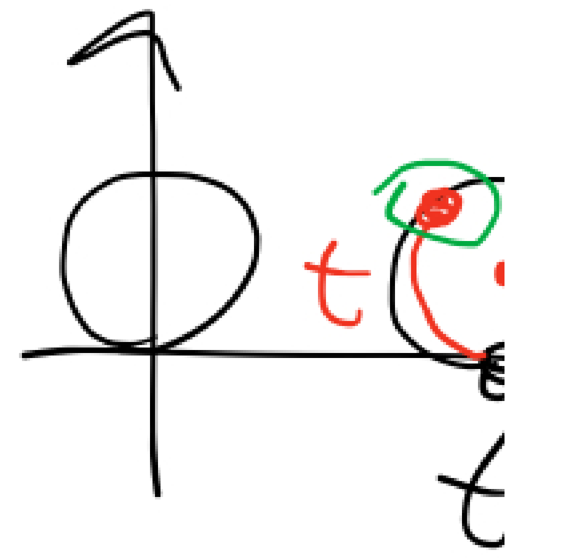
$$s = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{r^2 \cdot 1}\, dt = \int_0^{\pi} r\, dt = rt \Big|_0^{\pi} = \boxed{\pi r}$$

2) Find the arc length of one arch of the cycloid

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases} \quad t \in \mathbb{R}$$





$$s = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt = \int_0^{2\pi} \sqrt{1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}}\, dt$$

$$= \int_0^{2\pi} \sqrt{2-2\cos t}\, dt =$$

$$\sqrt{a^2} = |a| \;!$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{t}{2}}\, dt = \int_0^{2\pi} 2\left|\sin\frac{t}{2}\right| dt$$

$$\frac{1-\cos t}{2} = \sin^2\frac{t}{2}$$

$$= \int_0^{2\pi} 2\sin\frac{t}{2}\, dt = -4\cos\frac{t}{2}\Big|_0^{2\pi} =$$

$$= 8$$