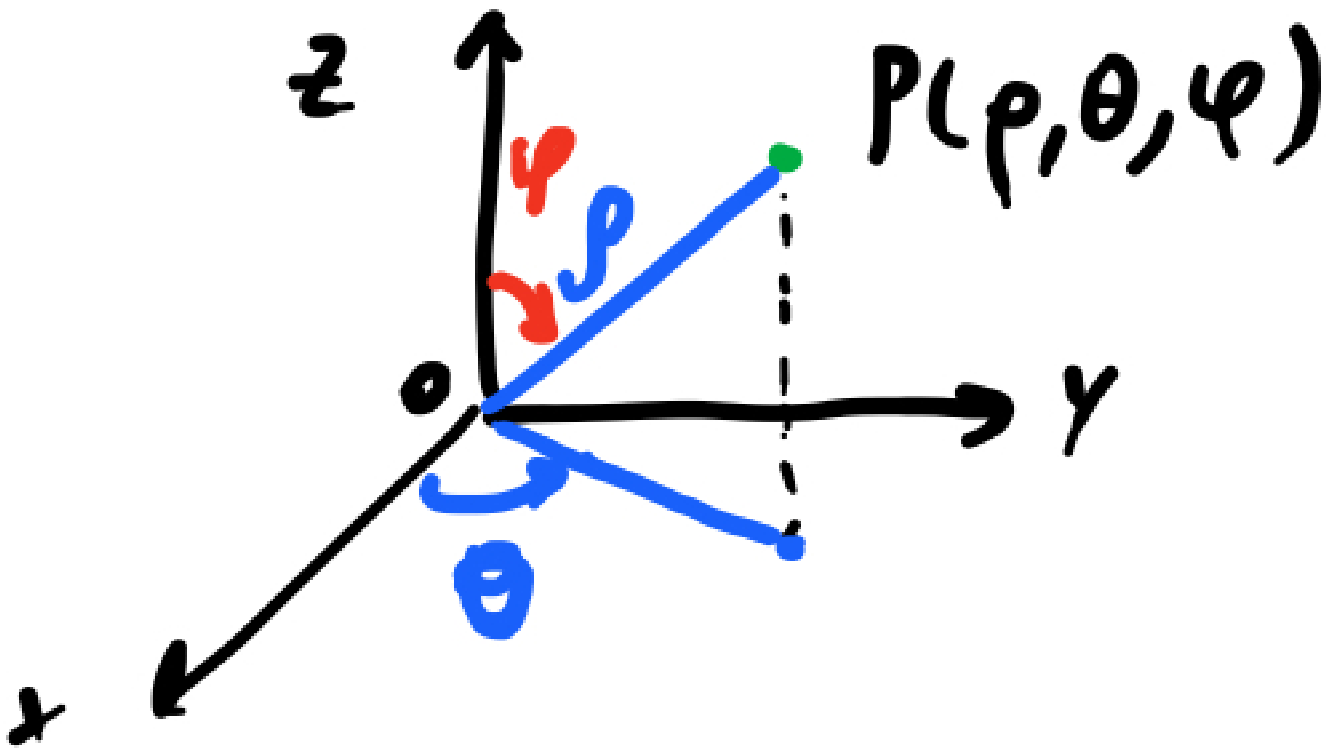
Triple integrals in spherical coordinates.



$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$

Spherical wedge

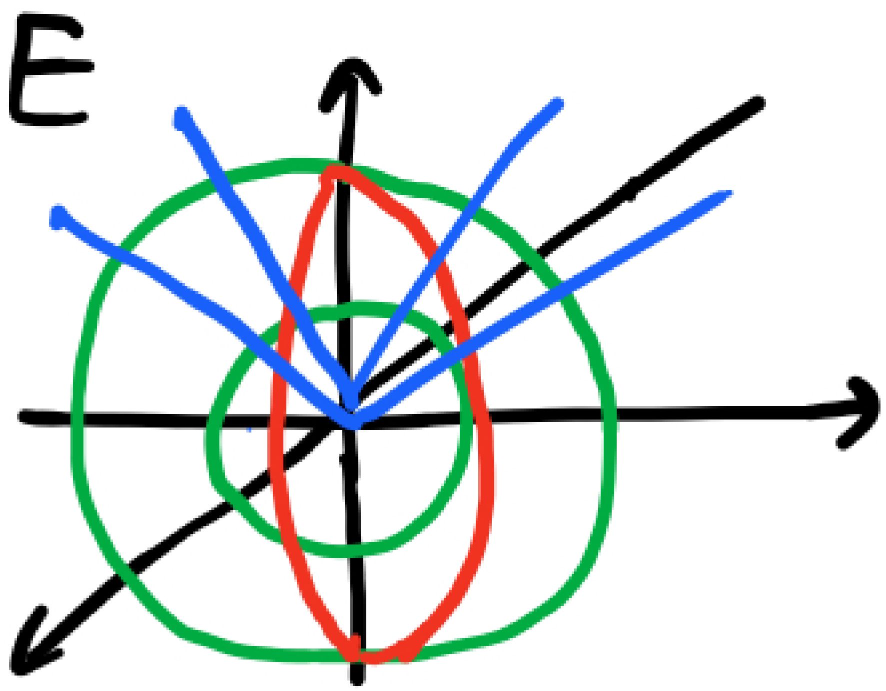
$$E = \left\{ (\rho,\theta,\varphi) \;\middle|\; \begin{array}{l} a \le \rho \le b \\ \alpha \le \theta \le \beta \\ c \le \varphi \le d \end{array} \right\}$$

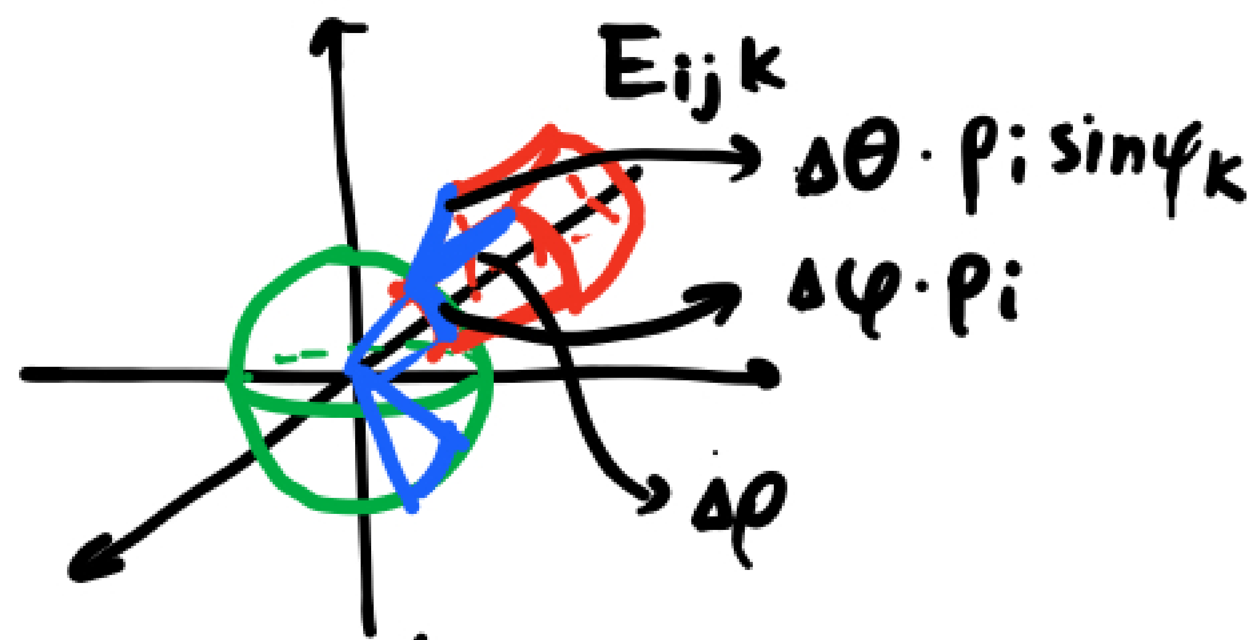
$b - a = \Delta\rho$

$\beta - \alpha = \Delta\theta$

$d - c = \Delta\varphi$

$(a \geq 0,\ \beta - \alpha \leq 2\pi,\ d - c \leq \pi)$





- Compute triple integral over E by subdividing into small spherical wedges E_{ijk} of volume $\Delta V_{ijk} \approx \Delta\rho\,\Delta\theta\,\Delta\varphi \cdot \underline{\rho_i^2 \sin\varphi_k}$

(wait until next time for a less vague proof)

$$\iiint_E f(x,y,z)\,\underline{dV} = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\underline{\rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi}$$

$V(B) =$

Example $\iiint_B 1 \, dV$ $\quad B = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$

$$\iiint_B 1 \, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} 1 \, \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi =$$

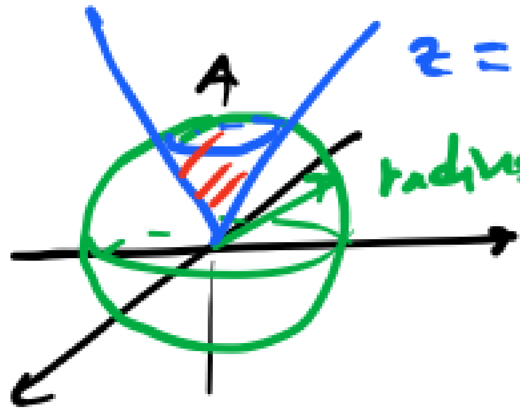
$$= \int_0^{\pi} \sin\varphi \, d\varphi \cdot \int_0^{2\pi} d\theta \cdot \int_0^{1} \rho^2 d\rho =$$

$$= -\cos\varphi \Big|_0^{\pi} \cdot 2\pi \cdot \frac{\rho^3}{3}\Big|_0^1 =$$

$$= 1-(-1) \cdot 2\pi \cdot \frac{1}{3} = \boxed{\frac{4\pi}{3}}$$

volume of unit ball

Example



$z = \sqrt{x^2+y^2}$

radius 2

$\rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta} = \rho\sin\varphi$

E

$$\iiint_E (x^2+y^2+z^2)\, dV$$

· express E

$$E = \left\{(\rho,\theta,\varphi) \;\middle|\; \begin{array}{l} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi/4 \end{array} \right\}$$

$$\iiint_E \underbrace{x^2+y^2+z^2}\, dV = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{2} \rho^2 \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi =$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^2 \rho^4\, d\rho \cdot \int_0^{\pi/4} \sin\varphi\, d\varphi =$$

$$= 2\pi \cdot \left.\frac{\rho^5}{5}\right|_0^2 \cdot \left.-\cos\varphi\right|_0^{\pi/4} =$$

$$= \boxed{2\pi \cdot \frac{32}{5} \cdot \left(1 - \frac{\sqrt{2}}{2}\right)}$$