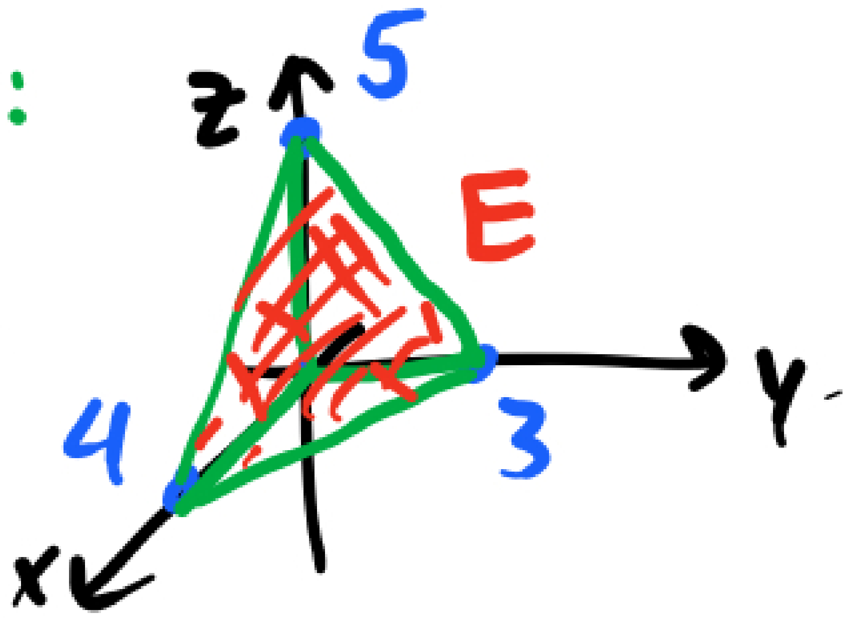
Property of triple integrals:

- $\iiint_E 1\, dV = V(E)$

- $\iint_R 1\, dA = A(R)$

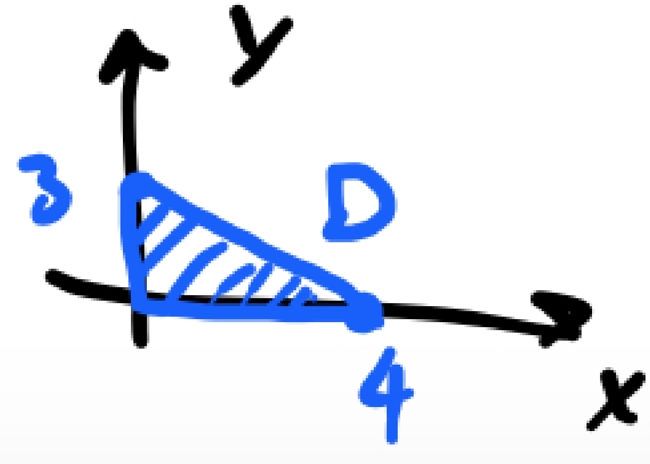
from last time :



$$\iiint_E x+y+z\, dV$$

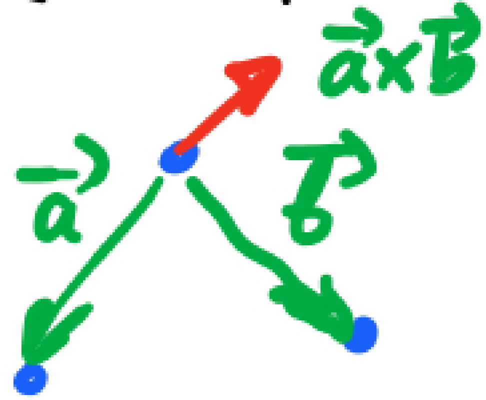
easier one---

$$\iiint_E 1\, dV$$



• express E • express D

$$E = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} (x,y)\in D \\ 0 \le z \le u_1(x,y) \end{array} \right\} \qquad D = \left\{ (x,y) \;\middle|\; \begin{array}{l} 0 \le x \le 4 \\ 0 \le y \le -\frac{3}{4}x+3 \end{array} \right\}$$



$(4,0,0)-(0,0,5)=$

$\vec{a} = (4,0,-5)$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 0 & -5 \\ 0 & 3 & -5 \end{vmatrix} =$$

$$= 15\hat{i} + 20\hat{j} + 12\hat{k}$$

$$15(x-4) + 20(y) + 12(z) = 0$$

$$15x + 20y + 12z = 60 \qquad /60$$

$$\underbrace{\frac{x}{4} + \frac{y}{3} + \frac{z}{5} = 1}$$

$$\Rightarrow z = \underbrace{5 - \frac{5x}{4} - \frac{5y}{3}}_{u_1(x,y)}.$$

$$\iiint_E 1\,dV = \iint_D \left[\int_0^{u_1(x,y)} 1\,dz\right] dA =$$

$$= \int_0^4 \left[\int_0^{-\frac{3}{4}x+3} \left[\int_0^{5-\frac{5x}{4}-\frac{5y}{3}} 1\,dz\right] dy\right] dx =$$

$$= \int_0^4 \int_0^{-3/4x+3} 5 - \frac{5x}{4} - \frac{5y}{3}\,dy\,dx =$$

$$= \int_0^4 5y - \frac{5xy}{4} - \frac{5y^2}{6} \Big|_0^{-3/4x+3} dx =$$

$$= \int_0^4 -\frac{15x}{4} + 15 + \frac{15}{16}x^2 - \frac{15}{4}x - \frac{5}{\cancel{6}_{2}}\left(\frac{\cancel{9}^{3}}{16}x^2 - \frac{18}{4}x + 9\right) dx$$

$$+ \frac{5 \cdot \cancel{18}^{3}}{\cancel{6} \cdot 4} x$$

$$= \int_0^4 \frac{15}{32}x^2 - \frac{15}{4}x + \frac{15}{2}\, dx =$$

$$= 15\left(\frac{x^3}{96} - \frac{x^2}{8} + \frac{x}{2}\right)\Big|_0^4 =$$

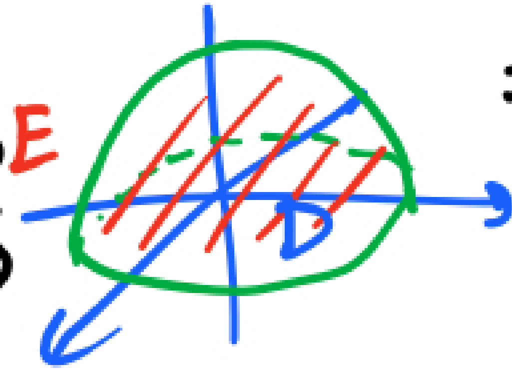
$$15\left(\underbrace{\frac{64}{96}}_{=2/3} - \cancel{\frac{16}{8}} + \cancel{\frac{4}{2}}\right) = \boxed{10}$$

<u>Another solution</u>: $\iiint_E 1\, dV = V(E) =$

$$= \frac{1}{3} A \cdot h = \frac{1}{3} \cdot 6 \cdot 5 = \boxed{10}$$

Example

$E = \left\{ (x,y,z) \,\middle|\, \begin{matrix} (x,y) \in D \\ 0 \le z \le \sqrt{\dots} \end{matrix} \right\}$



$z = \sqrt{1-x^2-y^2}$

top half of unit sphere

$$V(E) = \iiint_E 1 \, dV =$$

$$= \iint_D \left[\int_0^{\sqrt{1-x^2-y^2}} 1 \, dz \right] dA = \iint_D \sqrt{1-x^2-y^2} \, dA =$$

$$\overset{\text{polar}}{=} \int_0^{2\pi} \int_0^1 \underbrace{\sqrt{1-r^2}\; r}\, dr\, d\theta =$$

$$= 2\pi \int_0^1 \underbrace{r\sqrt{1-r^2}\, dr}_{-dx/2} =$$

$x = 1 - r^2$

$dx = -2r\,dr$

$$= \cancel{2}\pi \int_0^1 \sqrt{x}\, \frac{dx}{\cancel{2}} =$$

$$= \pi \, \frac{2}{3} x^{3/2} \Big|_0^1 = \boxed{\frac{2\pi}{3}}$$

Remark What if $f \neq 1$?

Use spherical coordinates!

Note type 2, type 3

$$E = \left\{ (x,y,z) \;\middle|\; \begin{array}{c} (x,z) \in D \\ u_1(x,z) \le y \le u_2(x,z) \end{array} \right\}$$

$$E = \left\{ (x,y,z) \;\middle|\; \begin{array}{c} (y,z) \in D \\ u_1(y,z) \le x \le u_2(y,z) \end{array} \right\}$$