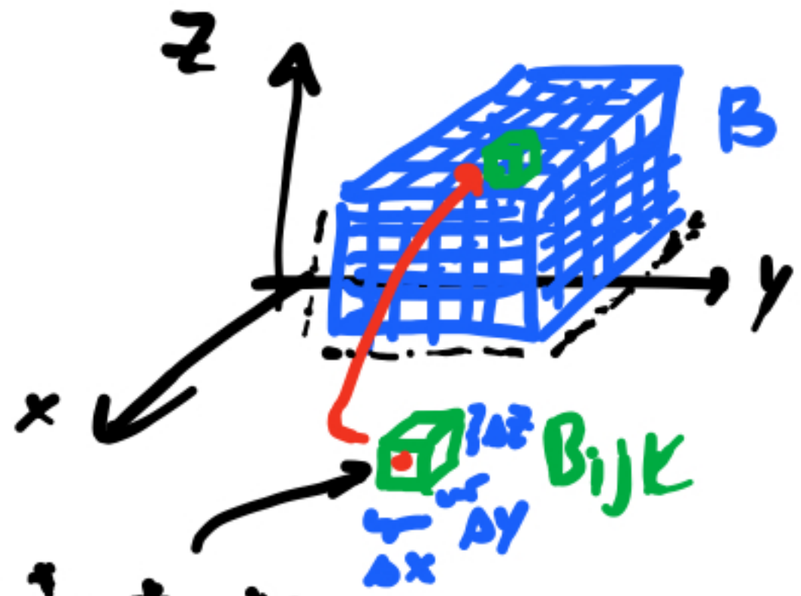
Triple Integrals

$$\iiint_E f(x,y,z)\,dV$$

E general domain

Domain: rectangular box

$$B = \{(x,y,z) \mid \begin{array}{l} a \le x \le b \\ c \le y \le d \\ r \le z \le s \end{array} \}$$



- divide B into sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

Volume of $B_{ijk} = \Delta V = \Delta x \Delta y \Delta z$.

- Riemann sum

sample pt

· Riemann Sum

$$\sum_{k=1}^{\ell}\sum_{j=1}^{m}\sum_{i=1}^{n} \underbrace{f(x_i^*, y_j^*, z_k^*)}_{\text{value at sample pt}} \Delta V$$

Define $\iiint_B f(x,y,z)\,dV = \lim_{n,m,\ell\to\infty}$ (Riemann Sum)

In practice, we will use iterated integrals:

Fubini's theorem

If f is continuous and $B = [a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \underbrace{\left[\int_c^d \underbrace{\left[\int_a^b f(x,y,z)\,dx\right]}_{\text{function of } y,z}\,dy\right]}_{\text{function of } z} dz$$

Remark We don't draw the graph of $f(x,y,z)$, because we would need to go into 4-dimensional space.

Example $\iiint_B xyz\,dV$, $B = [0,1]\times[-1,1]\times[-3,2]$.

There are 6 orders of integration wrt x,y,z

$$\iiint_B xyz\,dV = \int_{-3}^{2}\int_{-1}^{1}\int_{0}^{1} xyz\,dx\,dy\,dz =$$

$$= \int_{-3}^{2}\int_{-1}^{1} \frac{x^2yz}{2}\Big|_0^1 dy\,dz = \int_{-3}^{2}\int_{-1}^{1} \frac{yz}{2}\,dy\,dz =$$

$$= \int_{-3}^{2} \frac{y^2z}{4}\Big|_{-1}^{1} dz = \int_{-3}^{2} 0\,dz = 0.$$

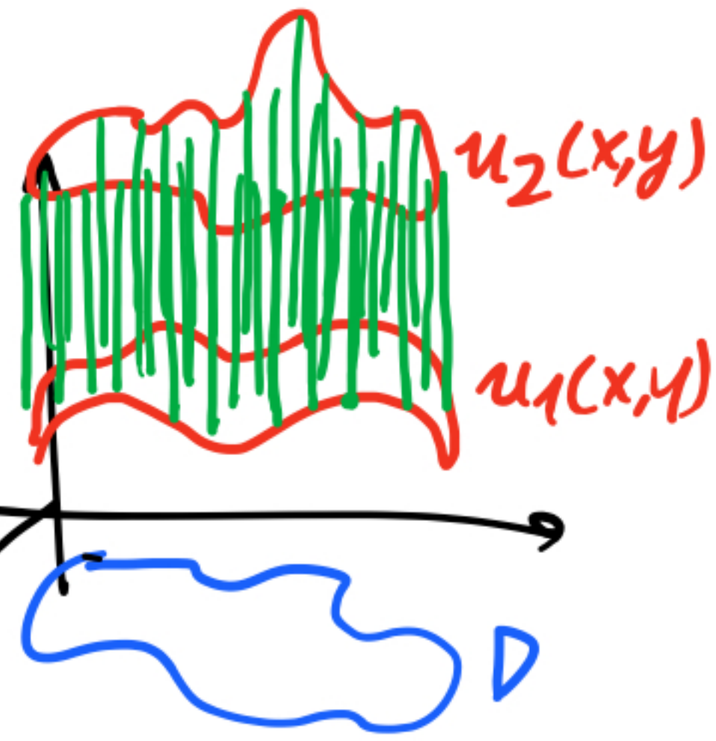
- We can define $\iiint_E f(x,y,z)\,dV$ similarly for a general bounded region E with nice boundary

Types of Regions

type 1

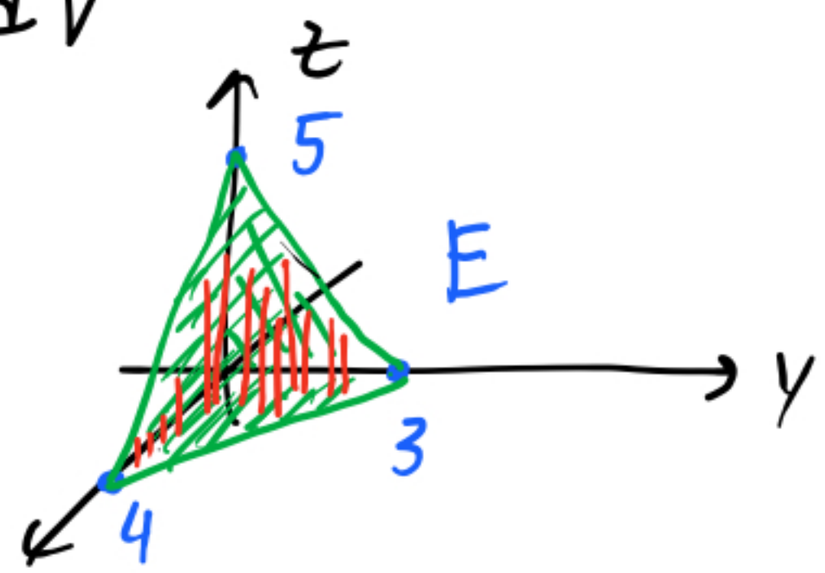
solid between two graphs

$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$



Then $$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

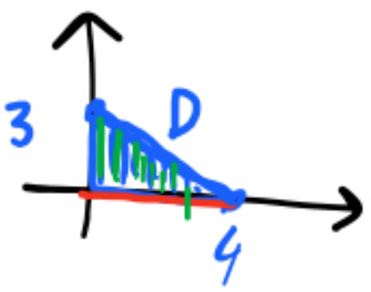
Example Find $\iiint_E (x+y+z)\,dV$



- Express E

$\{(x,y,z) \mid (x,y) \in D,\ 0 \le z \le u_2(x,y)\}$

u_2: graph of the face of tetrahedron



~~$u_2(x,y) = z = \frac{x}{4} + \frac{y}{3} = \frac{2y}{3}$~~ WRONG

$\frac{x}{4} + \frac{y}{3} + \frac{z}{5} = 1$ ↓ solve for z

- equation of the plane that passes through 3 given points

$$\iiint_E x+y+z\,dV = \iint_D \left[\int_0^{\frac{2y}{3}} x+y+z\,dz\right] dA = \text{next time!}$$