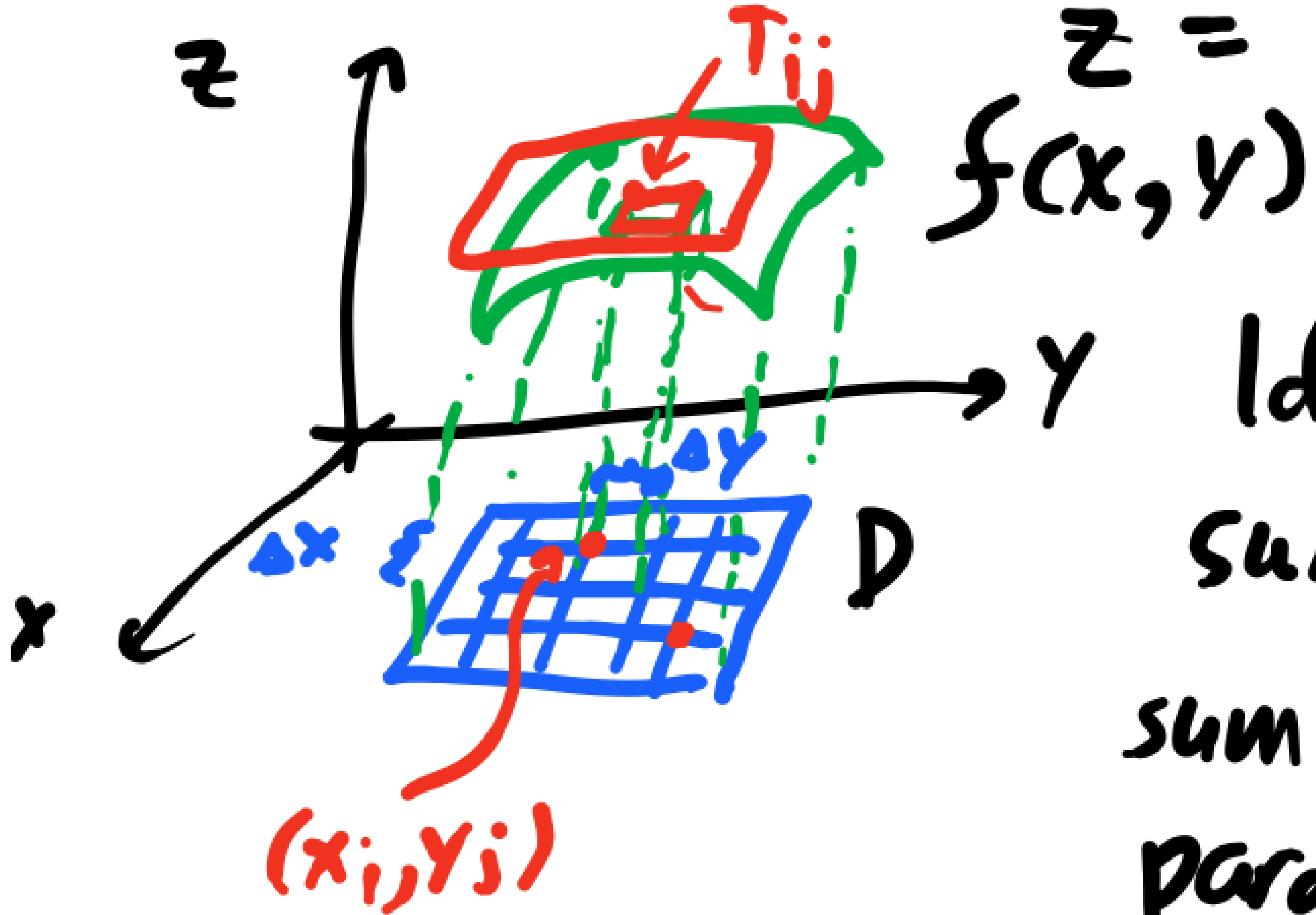
From last time:



Idea: approximate surface area with sum of areas of parallelograms (pieces of tangent planes)

Area of ΔT_{ij} :

$$\vec{a} = \Delta x \, \hat{i} + f_x(x_i, y_j)\, \Delta x \, \hat{k}$$

$$\vec{b} = \Delta y \, \hat{j} + f_y(x_i, y_j)\, \Delta y \, \hat{k}$$

Area of T_{ij} : $\| a \times b \|$

$$a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & f_x(\dots)\Delta x \\ 0 & \Delta y & f_y(\dots)\Delta y \end{vmatrix} =$$

$$= -\underbrace{f_x \Delta x \Delta y}\, \hat{i} - \underbrace{f_y \Delta x \Delta y}\, \hat{j} + \Delta x \Delta y\, \hat{k} =$$

$$= \Delta A \left(-f_x \hat{i} - f_y \hat{j} + \hat{k} \right).$$

$$\|a \times b\| = \sqrt{(-\Delta A f_x)^2 + (-\Delta A f_y)^2 + (\Delta A)^2} =$$

$$= \underbrace{\Delta A \sqrt{f_x^2 + f_y^2 + 1}}_{\text{area of red parallelogram } T_{ij}} .$$

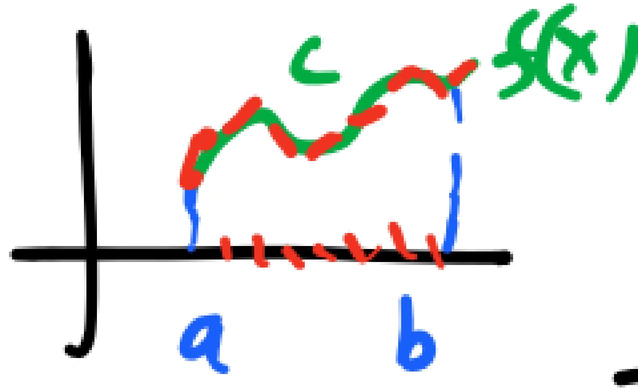
$$A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij} =$$

$$= \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{f_x^2 + f_y^2 + 1}\, \Delta A =$$

$$= \boxed{\iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA}$$

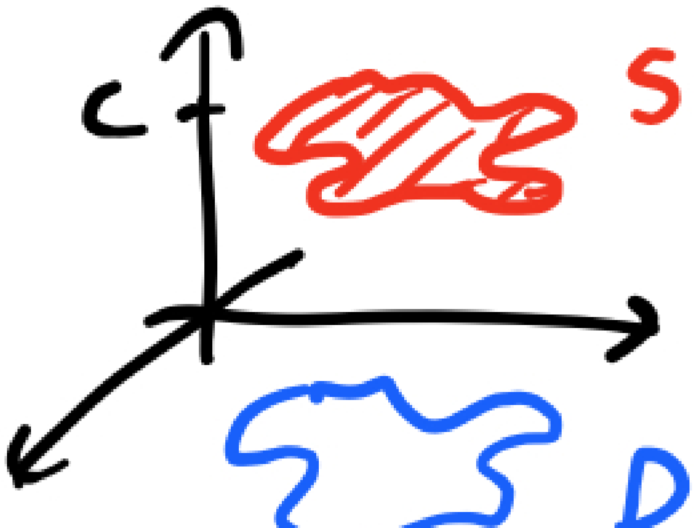
Remark

Compare with MAT136



$$\ell(C) = \int_a^b \sqrt{(f')^2+1}\,dx.$$

Example $z = \overset{f(x,y)}{C}$



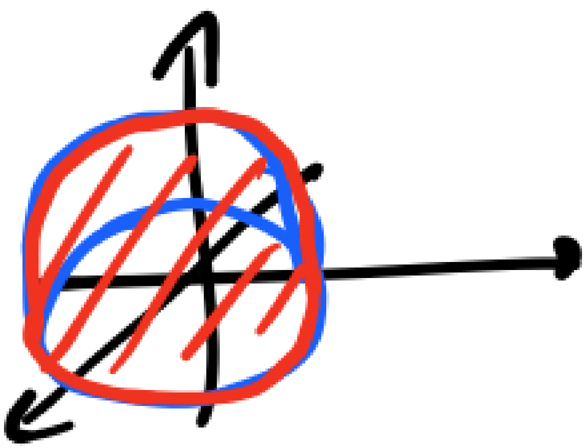
$$A(S) = \iint_D \sqrt{\underset{0}{\underset{\shortparallel}{f_x}}^2 + \underset{0}{\underset{\shortparallel}{f_y}}^2 + 1}\,dA =$$

$$= \iint_D \sqrt{1}\,dA = A(D)$$

✓.

Example $\quad z = \sqrt{1-x^2-y^2}$ – top half-sphere

$z^2 + x^2 + y^2 = 1$ – sphere



$$\frac{dz}{dx} = \frac{1\cdot(-\cancel{2}x)}{\cancel{2}\sqrt{1-x^2-y^2}}$$

$$\frac{dz}{dy} = \frac{-\cancel{2}y}{\cancel{2}\sqrt{1-x^2-y^2}}$$

$$\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1 = \frac{\cancel{4}x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2} + 1 =$$

$$= \frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2} + \frac{1-x^2-y^2}{1-x^2-y^2} =$$

$$= \frac{1}{1-x^2-y^2}.$$

$1-x^2-y^2.$

$$\iint_D \sqrt{\frac{1}{1-x^2-y^2}}\, dA \overset{\text{polar}}{=} \int_0^{2\pi}\int_0^1 \sqrt{\frac{1}{1-r^2}}\, r\,dr\,d\theta =$$

$$= \int_0^1 \int_0^{2\pi} r\sqrt{\frac{1}{1-r^2}}\, d\theta\, dr =$$

$$= \int_0^1 \left[\theta r \sqrt{\frac{1}{1-r^2}}\right]_0^{2\pi} dr =$$

$$= 2\pi \int_0^1 r\sqrt{\frac{1}{1-r^2}}\, dr$$

$r = \sin\theta$

$dr = \cos\theta\, d\theta$

$$= 2\pi \int_0^{\pi/2} \sin\theta \sqrt{\frac{1}{\underbrace{1-\sin^2\theta}_{\cos^2\theta}}} \cos\theta\, d\theta =$$

$$= 2\pi \int_0^{\pi/2} \sin\theta \frac{1}{\cos\theta} \cos\theta\, d\theta =$$

$$= 2\pi \int_0^{\pi/2} \sin\theta\, d\theta = 2\pi\left(-\cos\theta \Big|_0^{\pi/2}\right) =$$

$$= 2\pi$$

15.6. Triple Integrals

$$\int_a^b f(x)\,dx$$ ✓

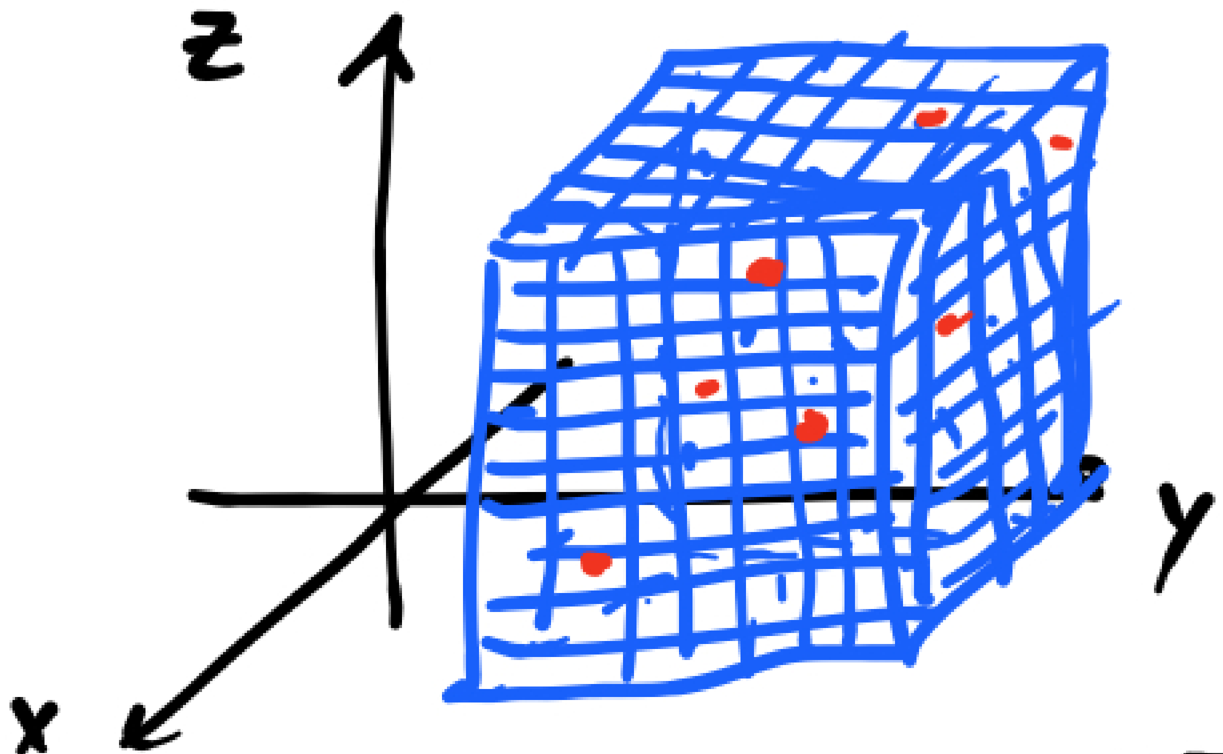
$$\iint_D f(x,y)\,dA$$ ✓

wait until Vector Calculus

$$\boxed{\iiint_E f(x,y,z)\,dV}$$?

Domain of $f(x,y,z)$: rectangular box

$$B = \left\{ (x,y,z) \;\middle|\; \begin{matrix} a \le x \le b \\ c \le y \le d \\ r \le z \le s \end{matrix} \right\}$$



- sample pts in small boxes

$$\sum\sum\sum f(x_i^*, y_j^*, z_k^*)\,\Delta V$$

$$\downarrow\; m,n,l \to \infty$$

$$\iiint f(x,y,z)\,dV$$