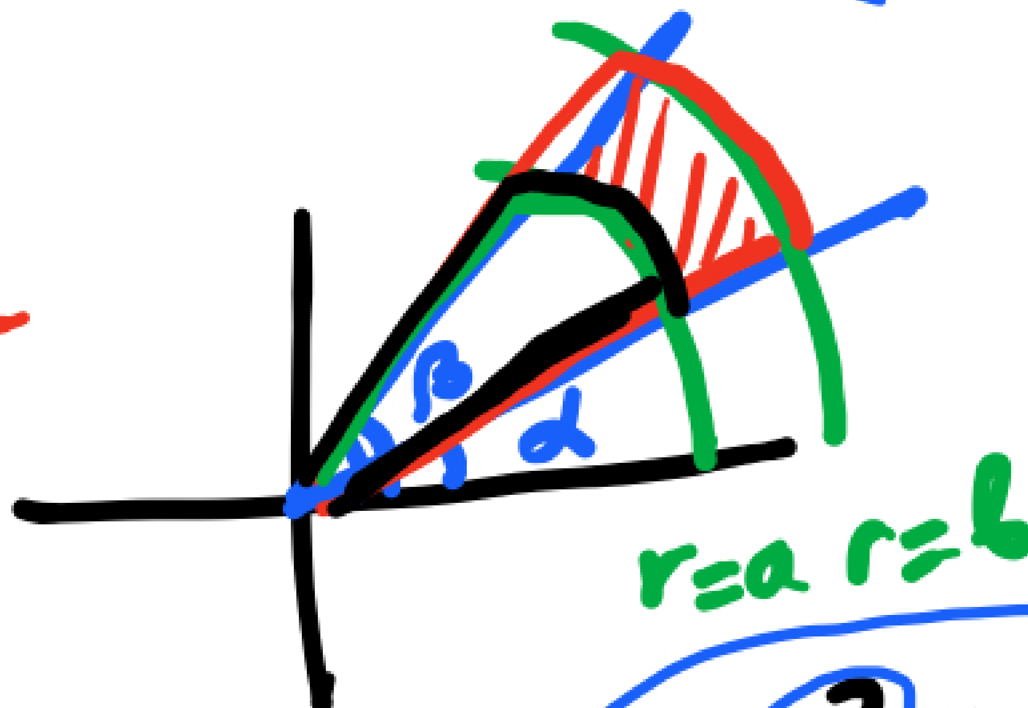
15.3. Double Integrals in Polar Coordinates

↳ makes it easy when the domain of integration is circular in nature

Exercise

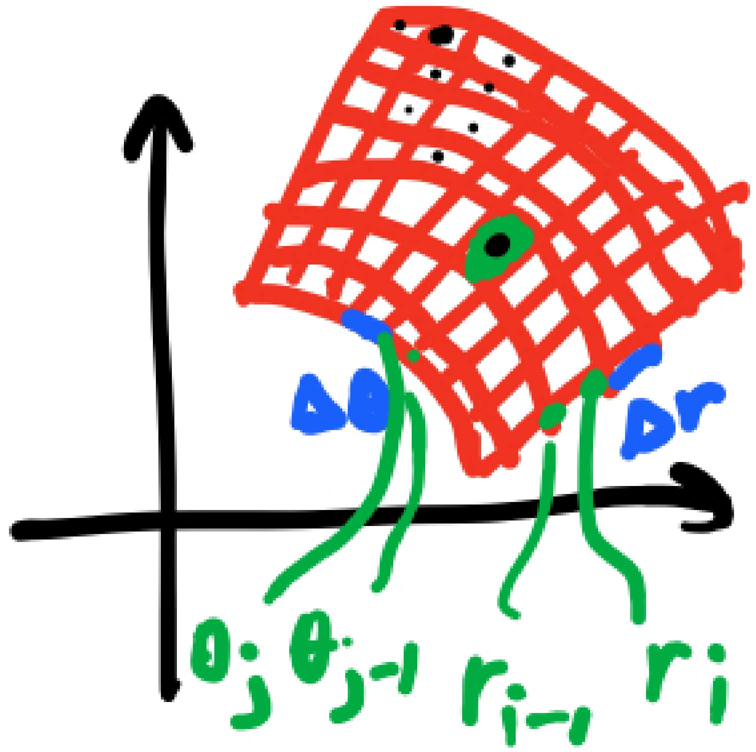


Area = difference of sectors =

$$= \frac{1}{2} b^2 (\beta - \alpha) - \frac{1}{2} a^2 (\beta - \alpha).$$

sector area

polar rectangle $R = \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$



To find $\iint_R f(x,y)\,dA$:

$$\Delta r = (b-a)/m$$

$$\Delta \theta = (\beta-\alpha)n$$

$$R_{ij} = \{(r,\theta) \mid r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$$

sample points: $r_i^* = \dfrac{r_{i-1}+r_i}{2}$

$$\theta_j^* = \frac{\theta_{j-1}+\theta_j}{2}$$

Area of $R_{ij} = \Delta A_i \overset{\text{Exercise}}{=} \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta$

$$= \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta = \frac{1}{2}(r_i + r_{i-1})\underbrace{(r_i - r_{i-1})}_{\Delta r}\Delta\theta =$$

$$= \underbrace{r_i^* \Delta r \Delta\theta}$$

Riemann sum:

$\downarrow$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,\Delta A_i =$$

$$= \sum\sum \underbrace{f(\ldots,\ldots)\, r_i^*}_{\lim\limits_{n,m\to\infty} = ?} \Delta r \Delta\theta \quad (*) =$$

If we write $g(r,\theta) = r f(r\cos\theta, r\sin\theta)$,

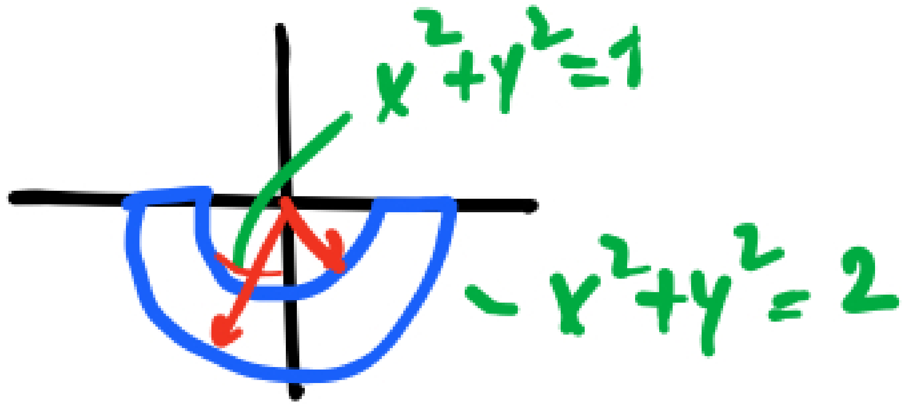
then $(*) = \sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r \Delta\theta$

$$\xrightarrow[n,m\to\infty]{} \int_{\alpha}^{\beta}\int_{a}^{b} g(r,\theta)\,dr\,d\theta =$$

$$= \boxed{\int_{\alpha}^{\beta}\int_{a}^{b} f(\overset{x}{r\cos\theta}, \overset{y}{r\sin\theta})\,\underbrace{r\,dr\,d\theta}}$$

Example Find $\iint_R (x^2-2y^2)\,dA$

$R =$



• describe R as a polar rectangle

$$R = \left\{ (r,\theta) \;\middle|\; \begin{array}{c} 1 \le r \le \sqrt{2} \\ \pi \le \theta \le 2\pi \end{array} \right\}$$

$$\iint_R x^2-2y^2\,dA = \int_\pi^{2\pi}\int_1^{\sqrt{2}} \left(r^2\cos^2\theta - 2r^2\sin^2\theta\right) \underline{r\,dr\,d\theta} =$$

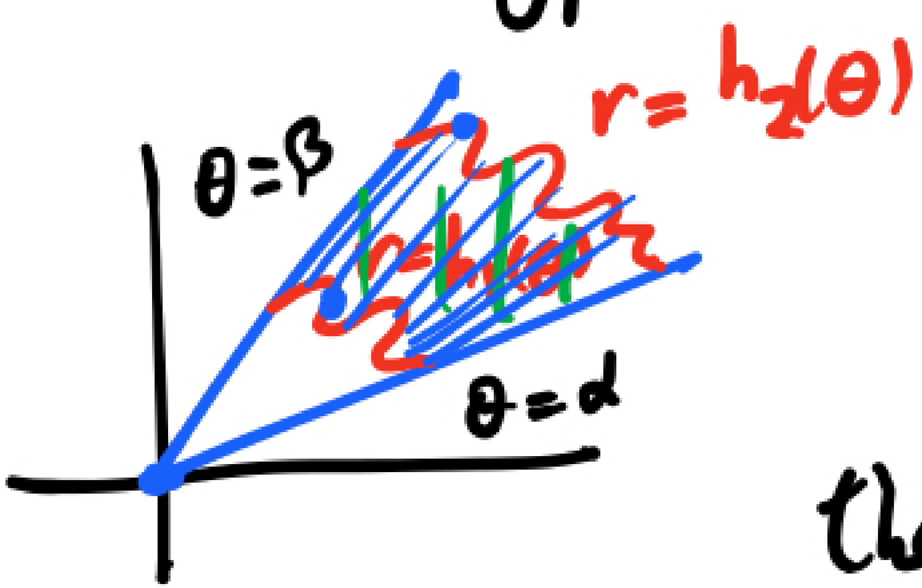
$$= \int_\pi^{2\pi}\left[\int_1^{\sqrt{2}} r^3\cos^2\theta - 2r^3\sin^2\theta \; dr\right] d\theta =$$

$$= \int_{\pi}^{2\pi} \left[\frac{r^4}{4}\cos^2\theta - \frac{r^4}{2}\sin^2\theta \Big|_1^{\sqrt{2}} \right] d\theta =$$

$$= \int_{\pi}^{2\pi} \frac{3}{4}\cos^2\theta - \frac{3}{2}\sin^2\theta \, d\theta = \ldots$$

Exercise.

"Type II" polar region:



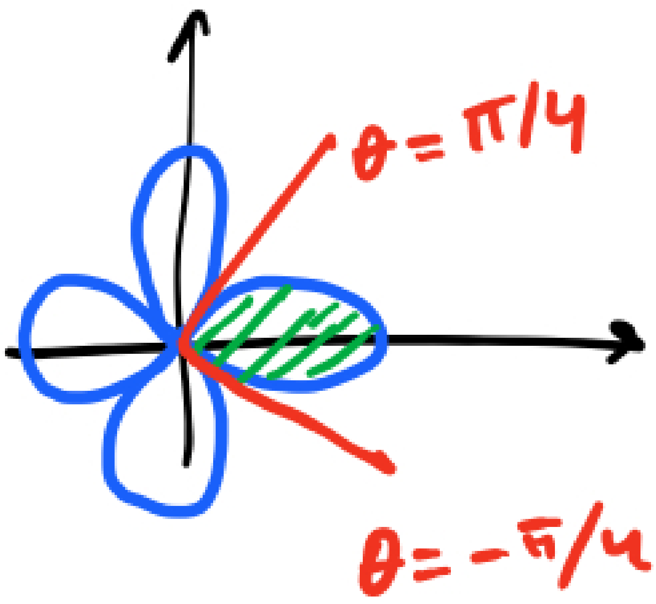
$$R = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} h_1(\theta) \le r \le h_2(\theta) \\ \alpha \le \theta \le \beta \end{array} \right\}$$

then $$\boxed{\iint_R f(x,y)\,dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Example

Find area enclosed by one loop of 4-leafed rose

$$r = \cos 2\theta$$



$$R = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} 0 \le r \le \cos 2\theta \\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \end{array} \right\}$$

$$A(R) \overset{\text{property}}{=} \iint_R 1 \, dA =$$

$$= \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} \underline{1 \; r} \, dr \, d\theta =$$

$$= \int_{-\pi/4}^{\pi/4} \frac{r^2}{2} \Big|_0^{\cos 2\theta} d\theta$$

Exercise

$$= \;\ldots\; = \boxed{\pi/8}$$