Properties of Double Integrals

linearity

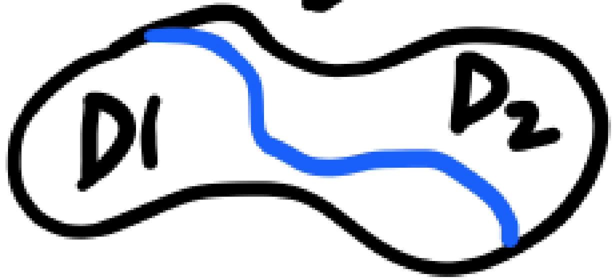
- $\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$

- $\iint_D c \cdot f(x,y)\, dA = c \iint_D f(x,y)\, dA$

- If $f(x,y) \geq g(x,y)$ in D,

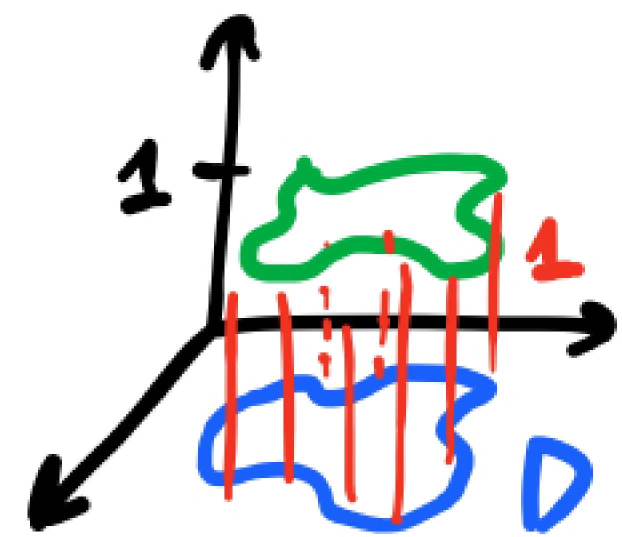
  then $\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$

D

- 

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

- $$\iint_D \overset{f(x,y)}{\overset{\shortparallel}{1}}\, dA = \text{Area}(D)$$



Example Find $\displaystyle\int_0^1 \left[\int_x^1 \sin(y^2)\,dy\right]dx$

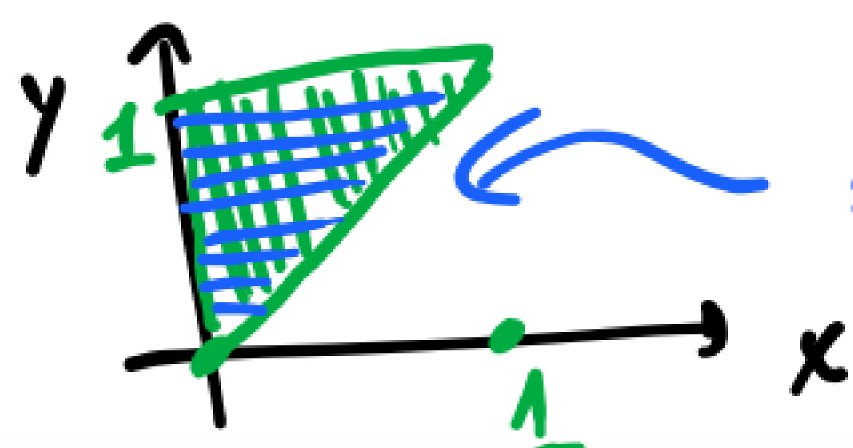
does not admit an antiderivative in "known" functions

Idea: switch the order of integration.

→ domain of integration?

$$D = \left\{ (x,y) \;\middle|\; \begin{array}{l} 0 \le x \le 1 \\ x \le y \le 1 \end{array} \right\}$$ from the integral

Draw D:



both Type I and Type II

$$D = \left\{ (x,y) \;\middle|\; \begin{matrix} 0 \le x \le y \\ 0 \le y \le 1 \end{matrix} \right\}$$

Then
$$\int_0^1 \int_x^1 \sin y^2 \, dy \, dx = \iint_D \sin y^2 \, dA$$

$$= \int_0^1 \left[\int_0^y \underbrace{\sin(y^2)}_{\text{doesn't depend on } x} \, dx \right] dy =$$

$$= \int_0^1 \left[x \sin(y^2) \Big|_0^y \right] dy =$$

$$= \int_0^1 \left[y \sin(y^2) - 0 \sin(y^2) \right] dy =$$

$$= \int_0^1 \underline{y \sin(y^2)}\, dy =$$

$$= -\frac{1}{2}\cos(y^2)\Big|_0^1 =$$

$$= -\frac{1}{2}(\cos 1) -$$

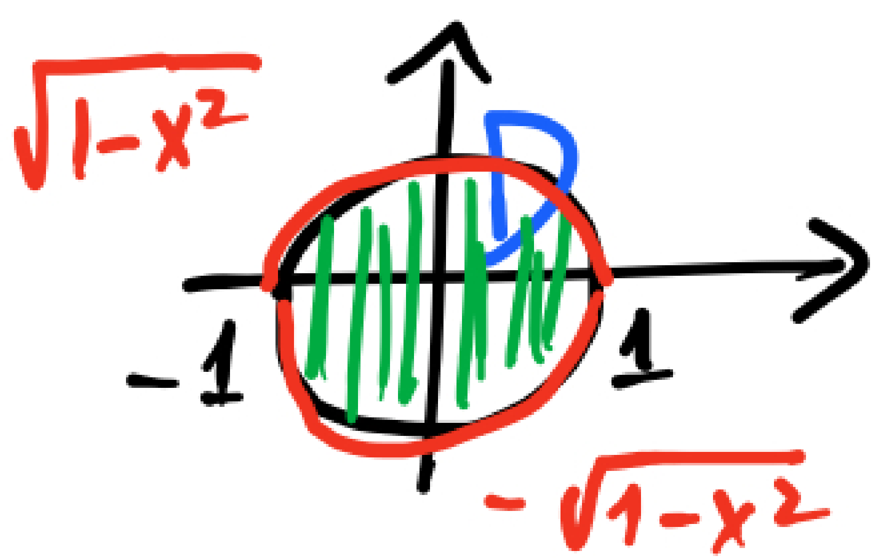
$$\left(-\frac{1}{2}\cos 0\right) =$$

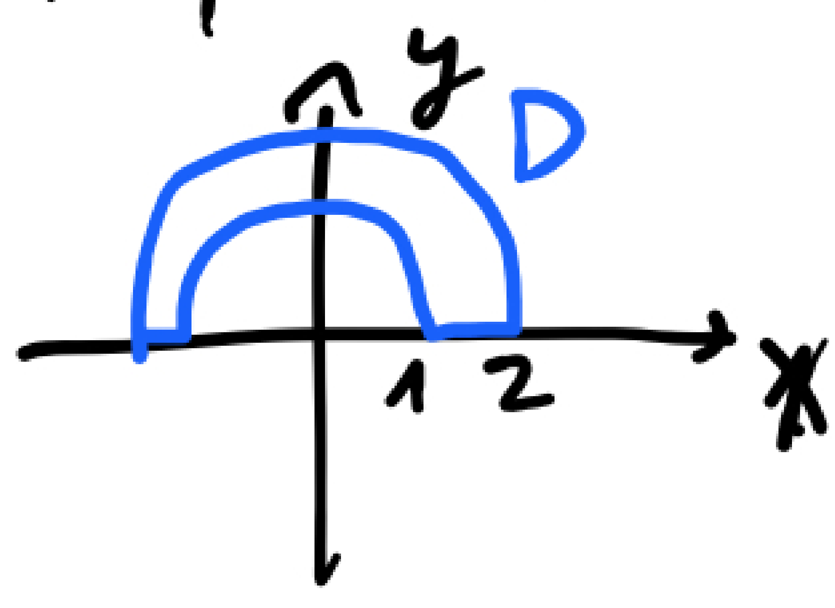
$$= \underline{\frac{1}{2}(1 - \cos 1).}$$

15.3. Double Integrals in Polar Coordinates.

Suppose we want to compute

$$\iint_D f(x,y)\,dA \quad , \text{ where } D$$





Example $\displaystyle\iint_D x^{2}\cancel{y}\, dA$, D - unit disk.

$$\int_{-1}^{1}\left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^{2}\cancel{y}\, dy\right]dx = \int_{-1}^{1}\left[x^2 y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\right]dx$$

painful to calculate

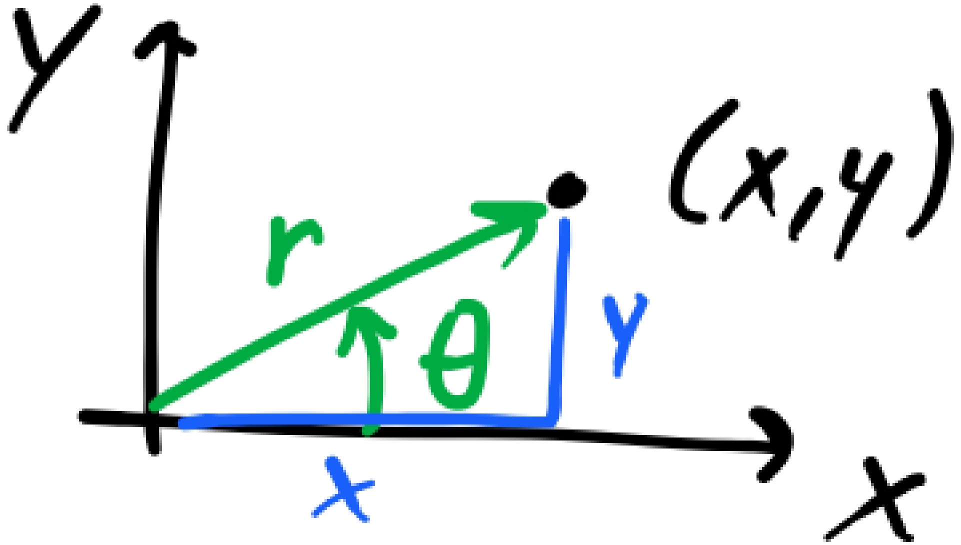
Solution: use polar coordinates.

e.g. unit disk $= \left\{ (r,\theta) \,\middle|\, \begin{array}{l} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{array} \right\}$

- How to go from double integral in xy to double integral in $r\theta$?

$$\iint_D x^2\,dA = \int_?^? \int_?^? (r\cos\theta)^2 \; ? \; dr\,d\theta$$

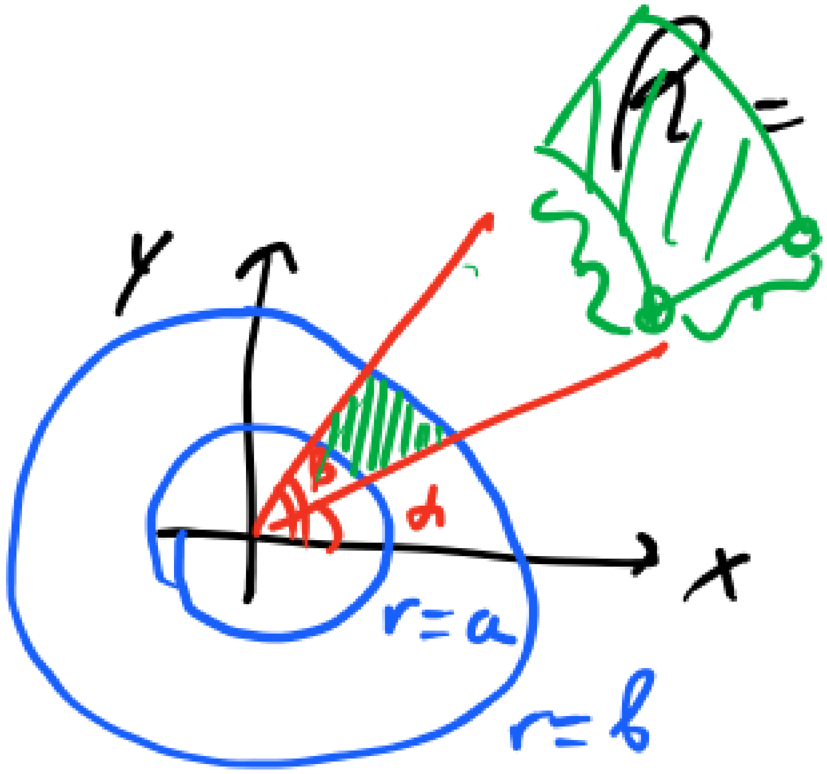
Recall:



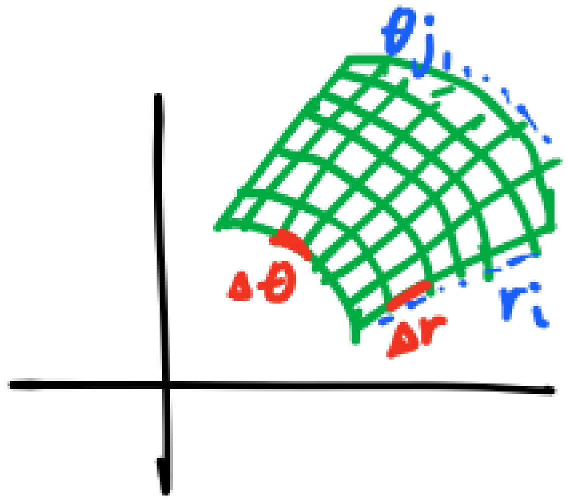
$$\boxed{\begin{aligned} r^2 &= x^2 + y^2 \\ x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

Define : polar rectangle

$$R = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} a \le r \le b \\ \alpha \le \theta \le \beta \end{array} \right\}.$$



To compute $\iint_R f(x,y)\,dA$



$$\Delta r = \frac{b-a}{m}$$

$$\Delta \theta = \frac{\beta-\alpha}{n}$$

$$R_{ij} = \left\{ (r,\theta) \;\middle|\; \begin{array}{l} r_{i-1} \le r \le r_i \\ \theta_{j-1} \le \theta \le \theta_j \end{array} \right\}$$

Area $R_{ij} = \Delta A_{ij} \overset{!}{=} \underbrace{\qquad\qquad}_{\text{Exercise}}$