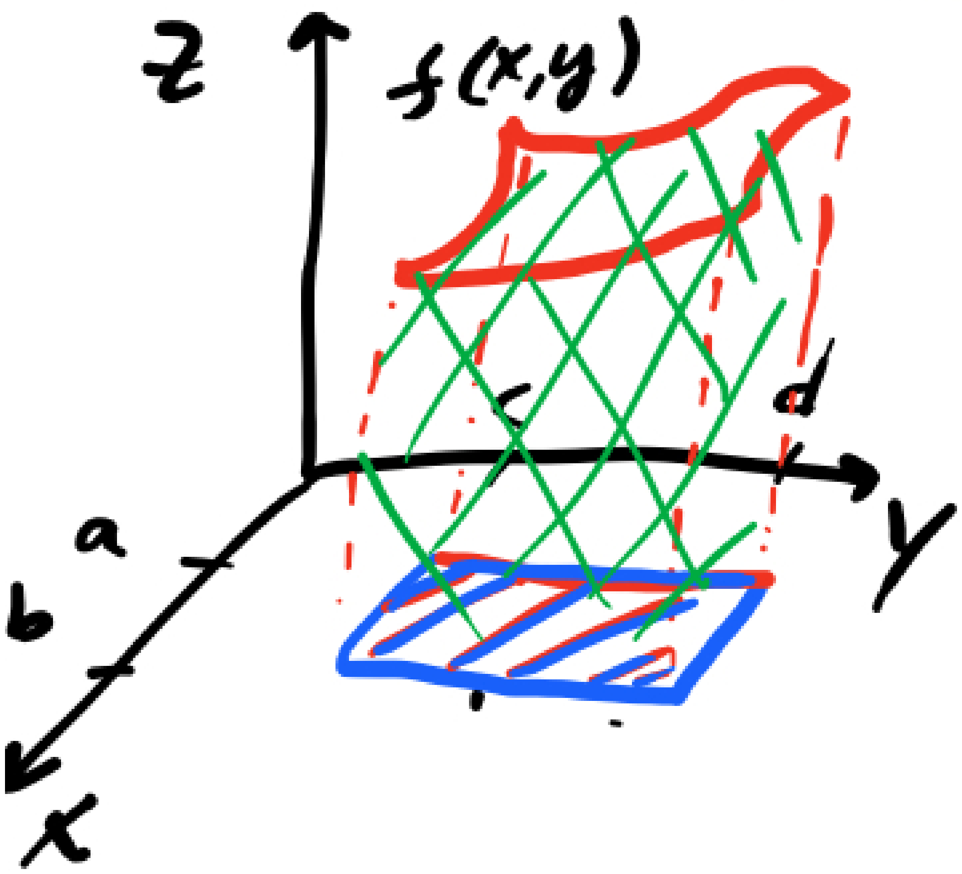
Double Integrals over Rectangles.



Setup: $f(x,y)$

Domain of $f(x,y)$:

$$R = [a,b] \times [c,d]$$

$$= \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} a \le x \le b \\ c \le y \le d \end{array} \right\}$$

$z = f(x,y)$ — graph of $f(x,y)$

Double integral $\iint\limits_R f(x,y)\, dA$ is

the volume of the solid S under $f(x,y)$

$$S = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 0 \le z \le f(x,y) \\ (x,y) \in R \end{array} \right\}$$

Construction:

$$\Delta x = \frac{b-a}{m}$$

$$\Delta y = \frac{d-c}{n}$$

sample points $(x_i^*, y_j^*) \in R_{ij}$

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

Let $\Delta A = \Delta x \cdot \Delta y$ – area of R_{ij}

Riemann sum

$$\sum_{j=1}^{n} \sum_{i=1}^{m} f(x_i^*, y_j^*) \cdot \Delta A$$

total volume of the collection of 3d rectangles

Definition

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\Big[\underbrace{\sum\sum f(\ldots)\,\Delta A}_{\text{Riemann sum.}}\Big]$$

Example

$$R = [-1,1] \times \cancel{[-3,3]}$$

$$[-1,1]$$

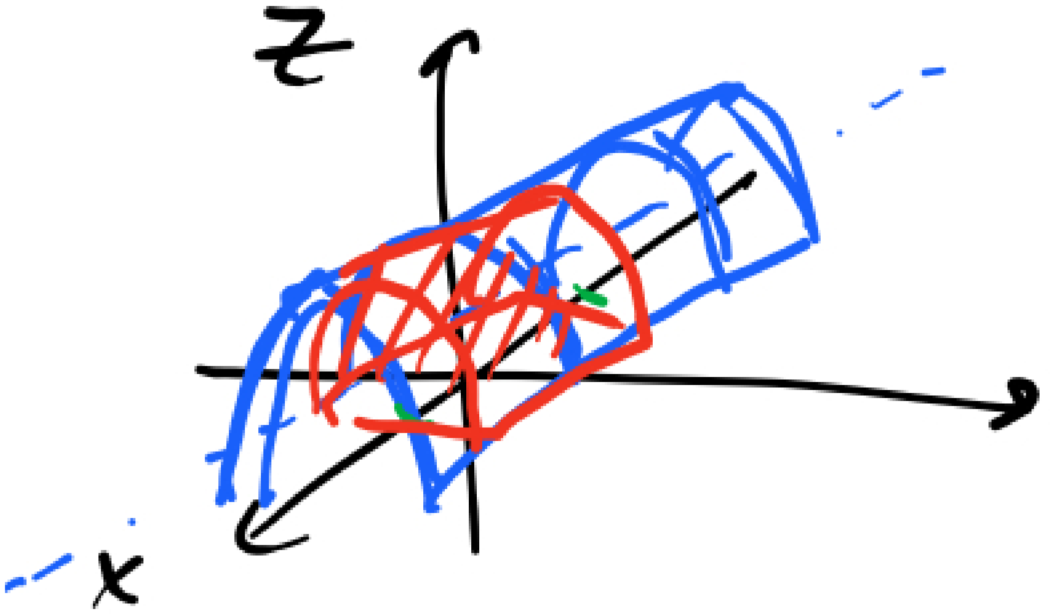
find

$$\iint_R \underbrace{\sqrt{1-y^2}}_{f(x,y)}\,dA.$$

x is free

$$z = \sqrt{1-y^2} \Rightarrow z^2 + y^2 = 1$$

– half-cylinder



$$\iint_R \sqrt{1-y^2}\, dA = \frac{1}{2}(\text{volume of the cylinder}) = \frac{1}{2}\pi 1^2 \cdot 2 = \boxed{\pi}.$$

How to compute double integrals?

→ use iterated integrals

<u>Fubini's theorem</u> If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx =$$

$$= \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy.$$

iterated integrals.

Example

$$I = \iint_R (xy - x^3)\, dA \qquad R = [-1,2] \times [3,4].$$

Exercise Apply Fubini to the example before

(i) $$I \overset{\text{Fubini}}{=} \int_{-1}^{2} \left[\int_3^4 (\ldots)\, dy \right] dx =$$

$$= \int_{-1}^{2} \left[x\frac{y^2}{2} - x^3 y \Big|_3^4 \right] dx =$$

$$= \int_{-1}^{2} \left[(8x - 4x^3) - \left(\tfrac{9}{2}x - 3x^3\right) \right] dx =$$

$$= \int_{-1}^{2} \left[\tfrac{7}{2}x - x^3 \right] dx =$$

$$= \frac{7}{4}x^2 - \frac{x^4}{4} \Big|_{-1}^{2} =$$

$$= (7 - 4) - \left(\tfrac{7}{4} - \tfrac{1}{4}\right) =$$

$$= \underline{\underline{1.5}}\,.$$

$$I = \int_3^4 \left[\int_{-1}^{2} (\ldots)\, dx \right] dy$$

$$= \int_3^4 \left[\frac{x^2}{2} y - \frac{x^4}{4} \Big|_{-1}^{2} \right] dy$$

$$= \int_3^4 [\text{ Exercise! }]$$