Example Find distance between the origin (in $\mathbb{R}^3$) and the curve of intersection of $\underset{\text{plane}}{x+y+z=1}$ and (double) cone $z^2=x^2+y^2$.

$$
\begin{cases}
\text{minimize} & \text{dist}^2 = x^2+y^2+z^2 = f(x,y,z) \\
\text{s.t.} & \overbrace{x+y+z}^{g(x,y,z)}=1 \\
 & \underbrace{z^2-x^2-y^2}_{h(x,y,z)}=0
\end{cases}
$$

Method:

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h & \text{3 eq}^{\text{ns}} \\ g = 1 \\ h = 0. \end{cases}$$

(provided ∇g and ∇h are not parallel so they span a plane, i.e. $\nabla g \times \nabla h \neq 0$)

$$\star \begin{cases} (2x, 2y, 2z) = \lambda(1,1,1) + \mu(-2x, -2y, 2z) \\ x + y + z = 1 & (4) \\ z^2 - x^2 - y^2 = 0 & (5) \end{cases}$$

Exercise: check that pts with $\nabla g \times \nabla h = 0$ don't satisfy $\star$.

1 $2x = \lambda - 2\mu x \iff 2x(1+\mu) = \lambda$ 1

... $2y(1+\mu) = \lambda$ 2

... $2z(1+\mu) = \lambda$ 3

- if $1+\mu = 0 \Rightarrow \lambda = 0 \Rightarrow$

3 $\Rightarrow z = 0$

5 $\Rightarrow x^2+y^2 = 0$

$\iff x=0,\ y=0.$

4 $\Rightarrow 0 = 1$ $\times$.

• if $1+\mu \neq 0$ $\quad x = \lambda / 2(1+\mu)$, $y = \lambda / 2(1+\mu)$

$\Rightarrow x = y.$

(4) $\Rightarrow z = 1 - 2x$

(5) $\Rightarrow (1-2x)^2 = 2x^2$

$2x^2 - 4x + 1 = 0$

.... $\Rightarrow x = 1 \pm \frac{1}{\sqrt{2}}, \; y = 1 \pm \frac{1}{\sqrt{2}}.$

$z = 1 - 2x = -1 \mp \sqrt{2}$

two pts: A, B

$\Rightarrow \lambda = \cdots, \; \mu = \cdots$ Check!

$\text{dist}(A) = \sqrt{6+4\sqrt{2}} = 2+\sqrt{2}$

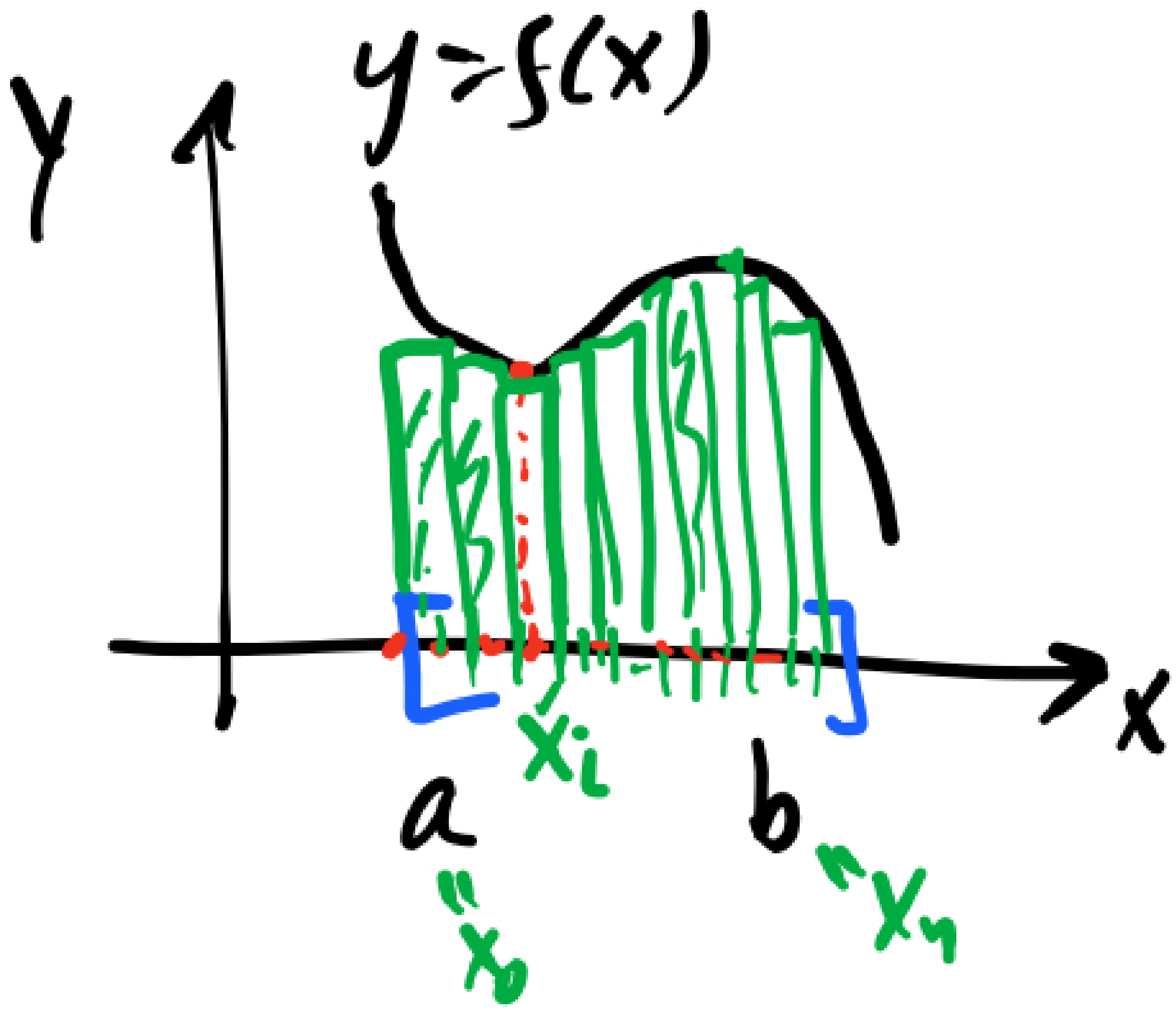
$\text{dist}(B) = \sqrt{6-4\sqrt{2}} = \underline{\underline{2-\sqrt{2}}}$

Multivariable Integration

15.1. Double Integrals over Rectangles.

- extend the idea of definite integral to double and triple integrals of two and three variables.

Review of Definite Integral



Set up: $f(x)$

$[a,b]$ – closed interval

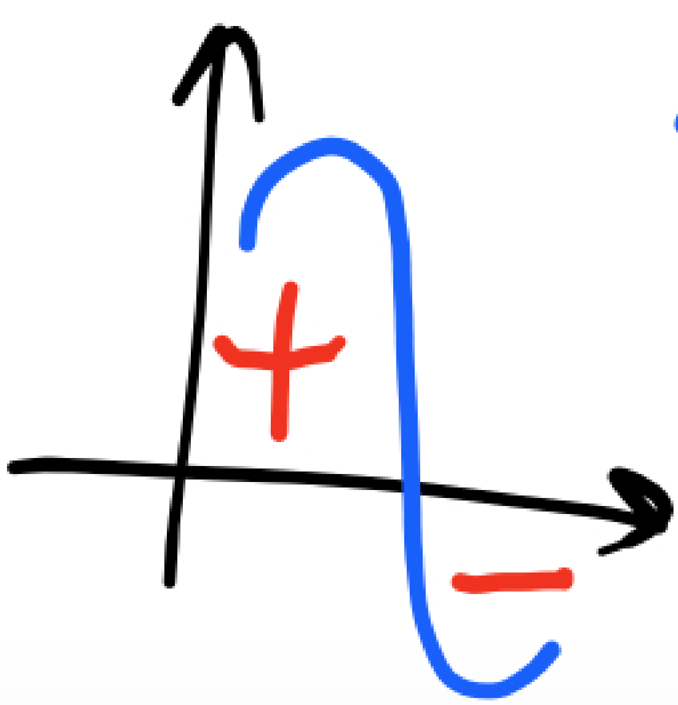
$\int_a^b f(x)\,dx$ – ?

length of subinterval : $\Delta x = \frac{b-a}{n}$

sample pts : $x_i^* \in [x_{i-1}, x_i]$

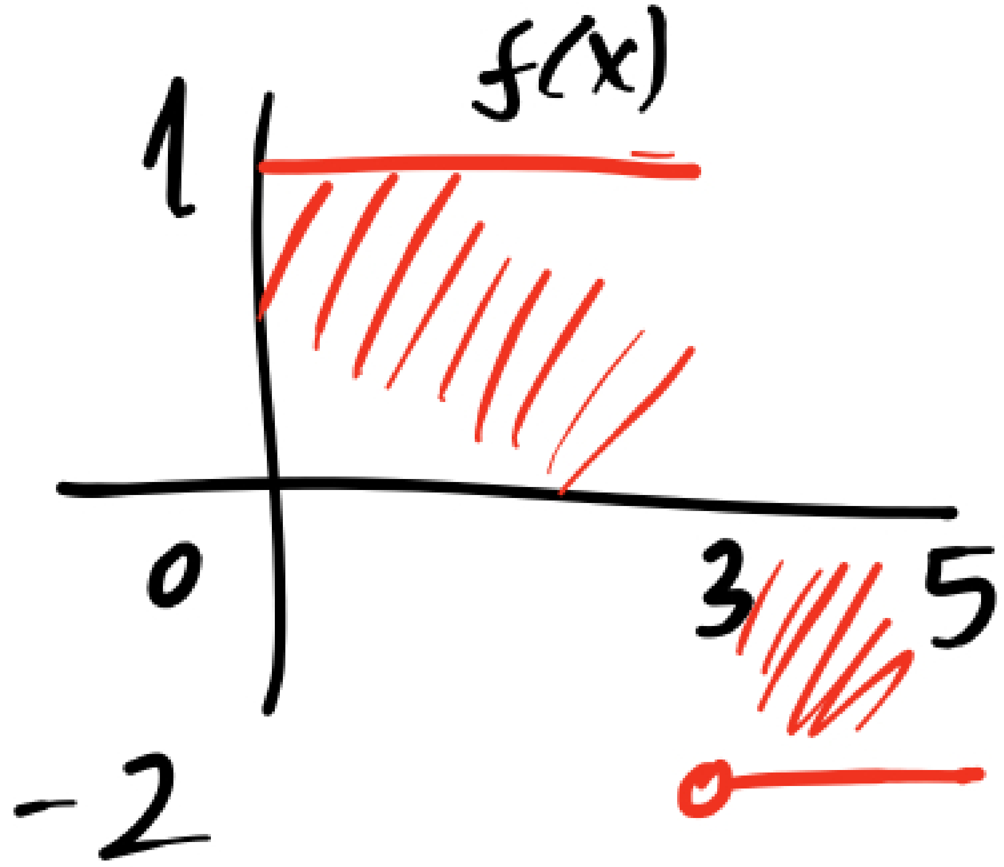
Riemann Sum : $\sum_{i=1}^{n} f(x_i^*)\Delta x$

$$\int_a^b f(x)\,dx \overset{\text{def}}{=} \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\,\Delta x\right)$$



- $\int_a^b f(x)\,dx$ represents the signed area under the graph of $f(x)$.

Example



$$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx = 3 - 4 = \underline{-1}.$$

Double integrals over Rectangles

Setup: $f(x,y)$

$$R = [a,b] \times [c,d]$$