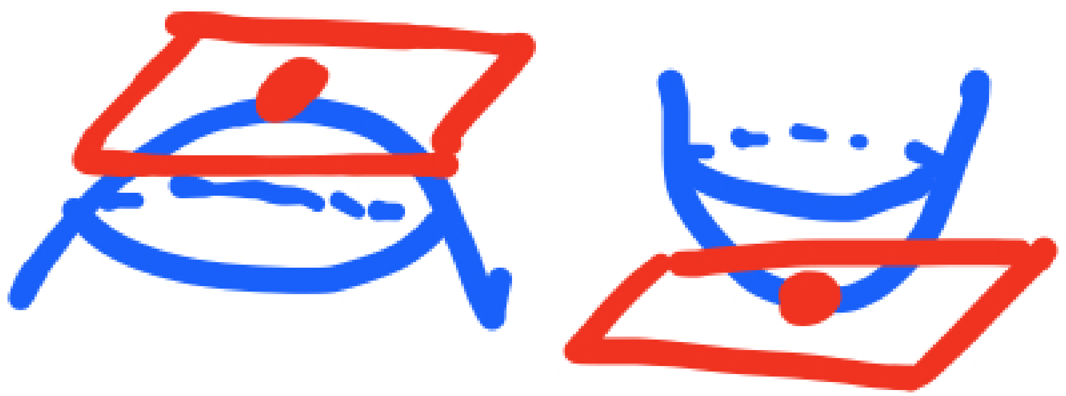
Last time:

• assuming f is differentiable

• if f has a local extrema at (a,b) $\Rightarrow$ tangent plane is horizontal $\Leftrightarrow$ $\nabla f(a,b) = \vec{0}$.



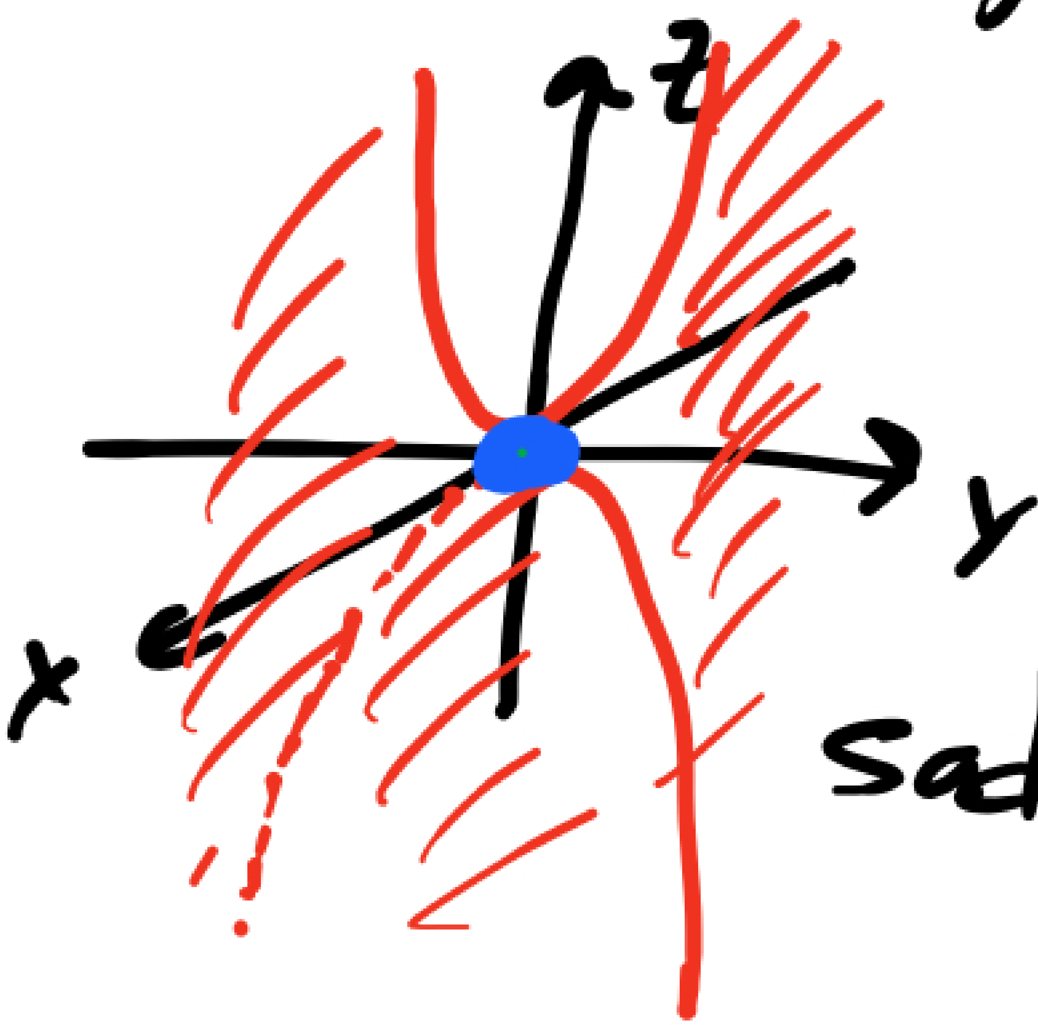
Def^n: We say (a,b) in the domain of f is a critical point if $\nabla f(a,b) = \vec{0}$ or if f_x or f_y DNE at (a,b).

will not focus on this

local max but f_x and f_y DNE

WARNING not every critical pt is a local extremum!

Example $f(x,y) = x^2 - y^2$



saddle

Crit pts:

$\nabla f = (2x, -2y)$

$\nabla f = \vec{0} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$

$(0,0)$ – only crit pt.

NOT a local max or min

Q: How to test whether a crit pt a local max, local min, or a saddle?

Reminder: (Calc 1) $\underbrace{\text{2nd derivative test}}$

assume f'' exists near c

c - critical pt $[f'(c) = 0]$

$\boxed{f''(c) > 0} \Rightarrow$ local min

$\boxed{f''(c) < 0} \Rightarrow$ local max

$\boxed{f''(c) = 0} \Rightarrow$ no info

2nd derivative test for functions of two variables

Suppose $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ are continuous near (a,b)

[by Clairaut's thm, $f_{xy} = f_{yx}$]

Suppose $\nabla f(a,b) = \vec{0}$ (i.e. (a,b) is a critical pt)

Let $D = D(a,b) = f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2$.

Then

$D>0$ & $f_{xx}(a,b)>0$	$D>0$ & $f_{xx}(a,b)<0$	$D<0$	$D=0$
$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$
local min	local max	"saddle"	no info

Notes: ① if $D > 0$, then

$$f_{xx} \cdot f_{yy} > (f_{xy})^2$$

$\Rightarrow$ if $f_{xx} > 0$, then $f_{yy} > 0$

(and if $f_{yy} > 0$, then $f_{xx} > 0$)

② D is called Hessian determinant.

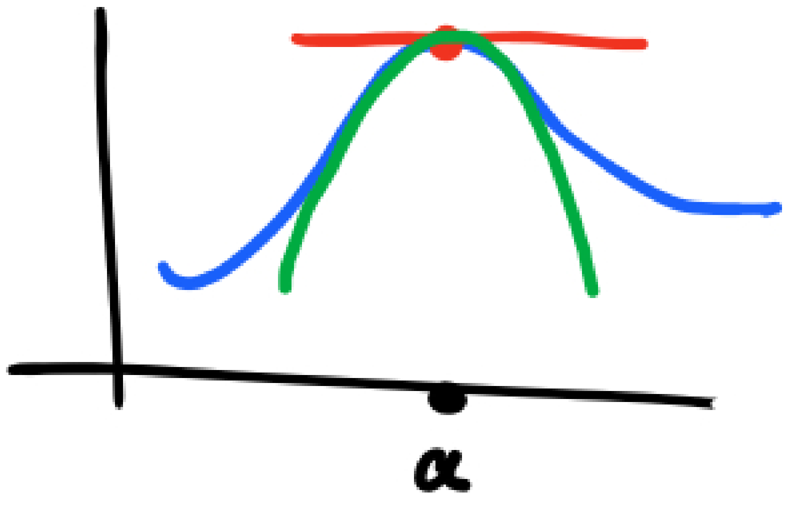
In general, Hessian matrix

is $H(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ (like $\nabla f = (f_x, f_y)$ "first derivative")

"second derivative"

and $D = \det H(f) = f_{xx} f_{yy} - (f_{xy})^2$

"Explanation" of the test:



$$f(x) \approx f(a) + \cancelto{0}{f'(a)}(x-a)$$

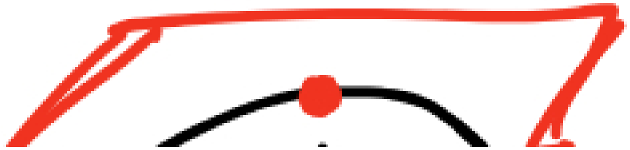
$(x \approx a)$

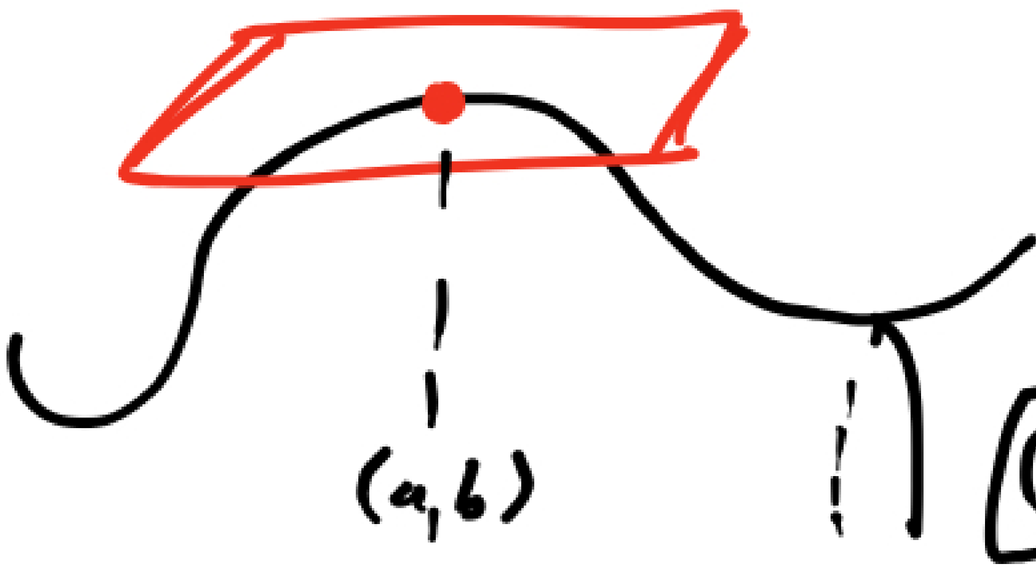
$$+ \frac{f''(a)}{2}(x-a)^2$$

"Taylor polynomial"

If $f''(a) < 0$, then $f(x) < f(a)$ for x near a

$\Rightarrow f(a)$ – local max.





$\Rightarrow f(a)$ – local max.

$[(x,y) \approx (a,b)]$

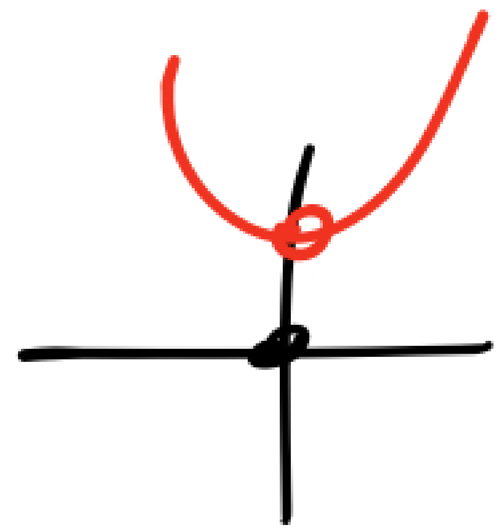
$$f(x,y) \approx f(a,b) + \cancelto{0}{f_x(a,b)}(x-a) + \cancelto{0}{f_y(a,b)}(y-b)$$

$$\left.\begin{aligned} &+ \frac{f_{xx}(a,b)}{2}(x-a)^2 \\ &+ f_{xy}(a,b)(x-a)(y-b) \\ &+ \frac{f_{yy}(a,b)}{2}(y-b)^2 \end{aligned}\right\} \text{quadratic terms}$$

linear algebra tells
the sign of this expression
[keyword: quadratic form]

Examples

(1) $f(x,y) = 1 + x^2 + y^2$



crit pts: $\nabla f = \vec{0}$

$(2x, 2y) = (0,0)$

$\iff (x,y) = (0,0).$

2^{ND} deriv test:

$H(f) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $D = 4 > 0$

$f_{xx} = 2 > 0$ $\Rightarrow$ local min.