Last time:

Significance of ∇f

(a) • gradient vector points in the direction of maximal increase of f

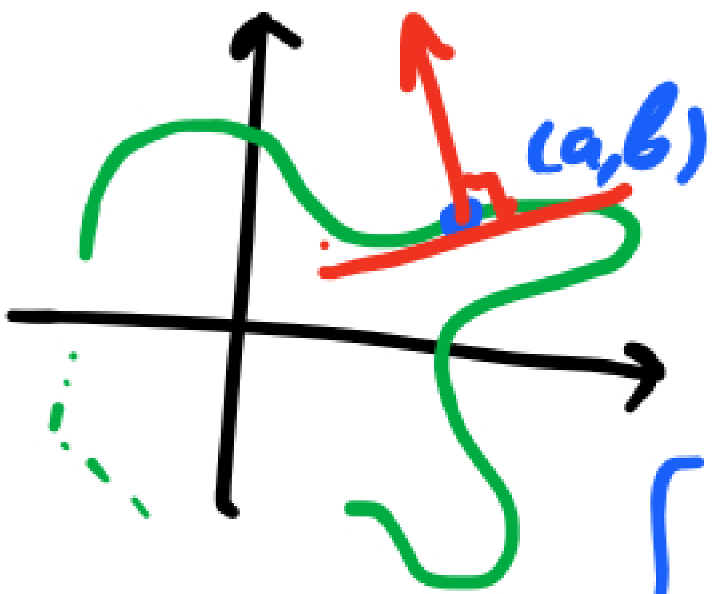
• the magnitude represents ~~how~~ the speed of increase

$$D_{\hat{u}} f = \| \nabla f \|$$

direction of max increase

increase

(b) ∇f gives normal vectors to level curves and level surfaces



$$\boxed{f(x,y)=c}$$

- Normal to the curve at (a,b) is given by $\nabla f(a,b)$

Suppose that y is a function of x. Then the slope of the tangent line

$$\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$ (implicit diff.)

Reminder:

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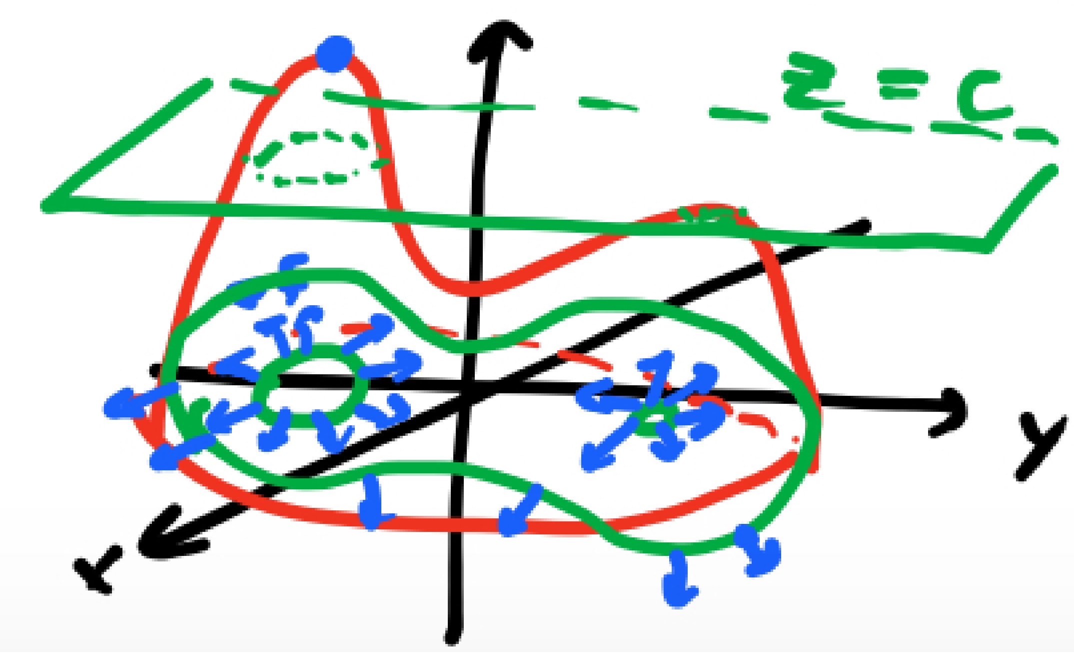
line with slope m has dir. vector (1,m)

$\Rightarrow$ slope of the normal line $= \dfrac{f_y}{f_x}$.

a direction vector of it is $\left(1, \dfrac{f_y}{f_x}\right) =$

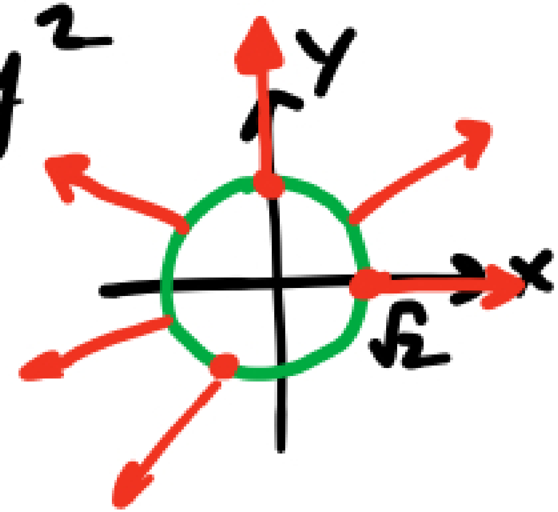
$$= \left(\frac{f_x}{f_x}, \frac{f_y}{f_x}\right) = \frac{1}{f_x}(f_x, f_y).$$

Upshot: we can take $(f_x, f_y) = \nabla f$ as a normal vector to the level curve.



Example

$$f(x,y) = x^2 + y^2$$



$$x^2 + y^2 = 2$$

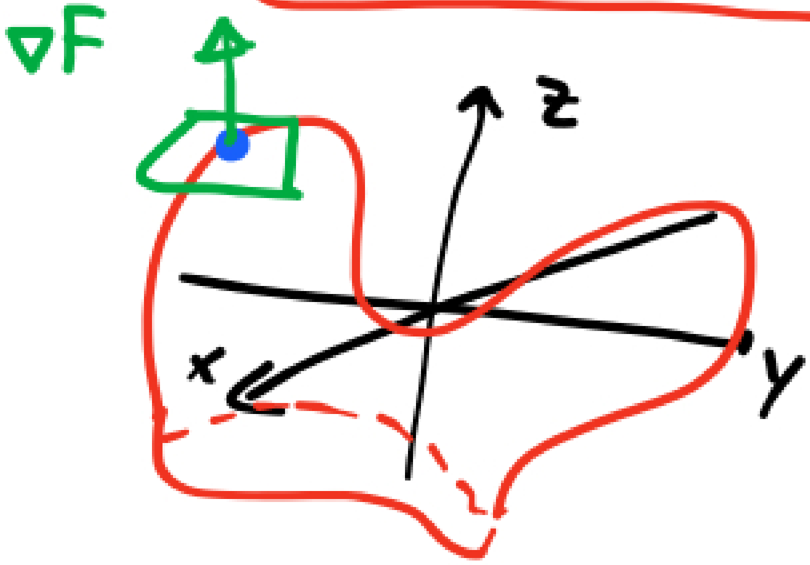
$$\nabla f(x,y) = (2x,\ 2y) =$$

$$= 2\,(x,y)$$

$$\nabla f(\sqrt{2}, 0) = (2\sqrt{2}, 0)$$

$$\nabla f(0, \sqrt{2}) = (0, 2\sqrt{2})$$

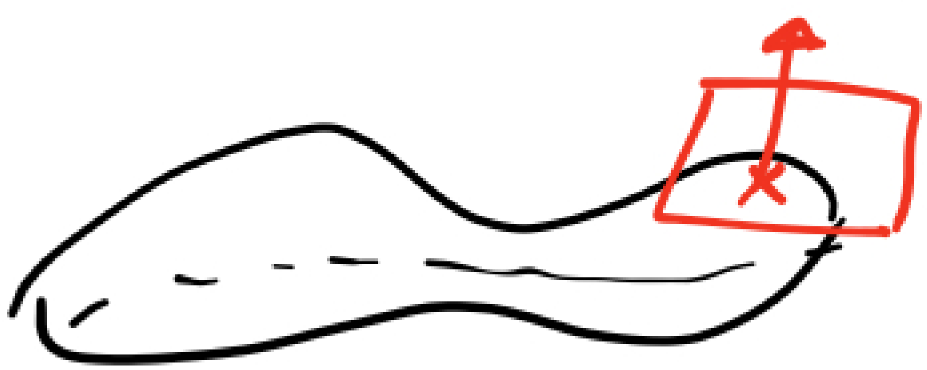
If $\boxed{F(x,y,z)=C}$



Claim: $\nabla F(a,b,c)$ is $\perp$ to the tangent plane to the surface $F(x,y,z)=C$ at (a,b,c).

Example Suppose $z=f(x,y)$.

We know $\vec{n} = (-f_x, -f_y, 1)$

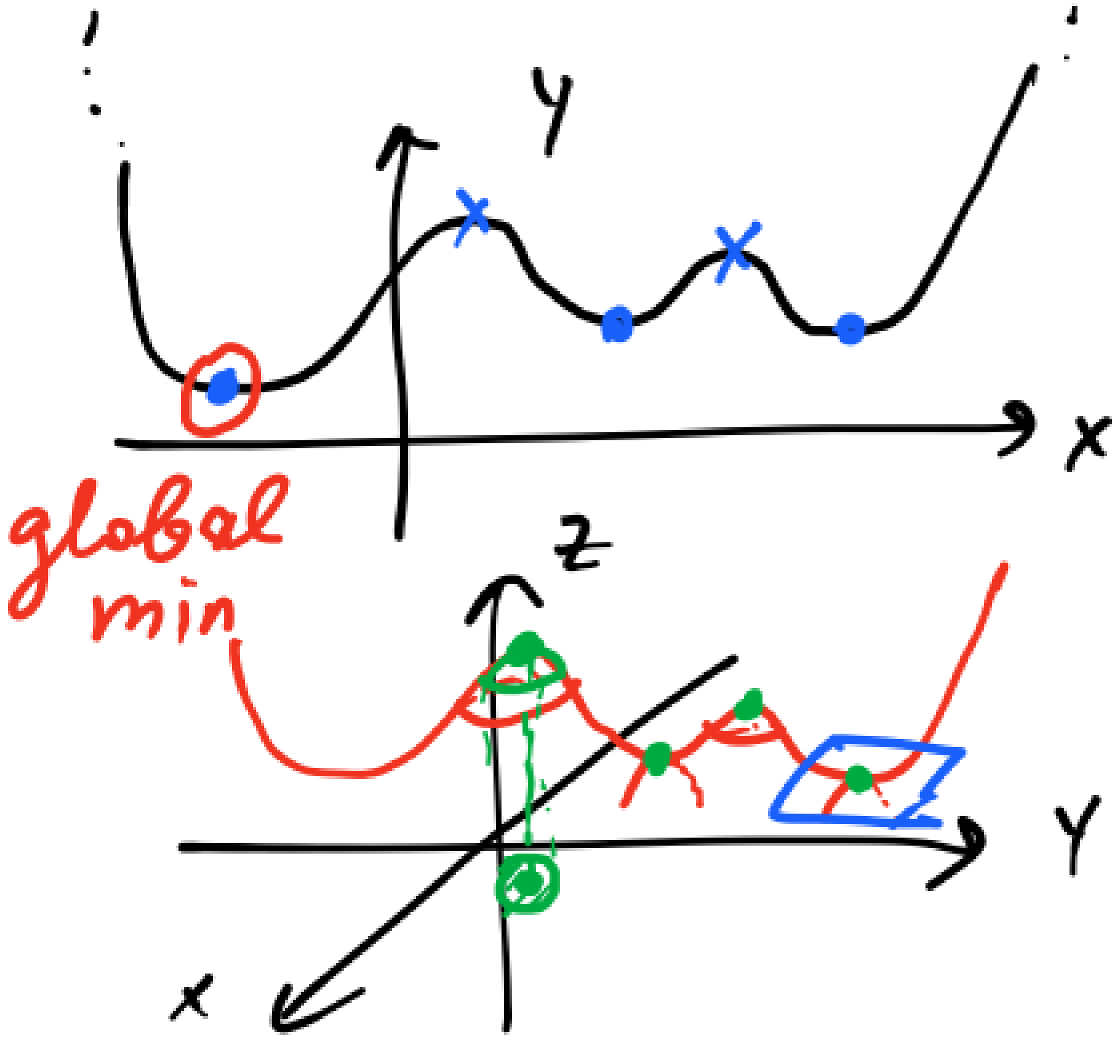


(from lecture on tangent planes)

$$\underbrace{z-f(x,y)}_{F(x,y,z)} \overset{+c}{=} C$$

$$\nabla F = (F_x, F_y, F_z) = (-f_x, -f_y, 1).$$

14.7. Maximum and minimum values.



- local minima $\Big\}$ local extrema
- local maxima

Def^n We say a value $f(a,b)$ is a **local minimum value** for $f(x,y)$ if

$f(a,b) \le f(x,y)$ for (x,y) near (a,b)

- - - - — **local max** - - - (similar)

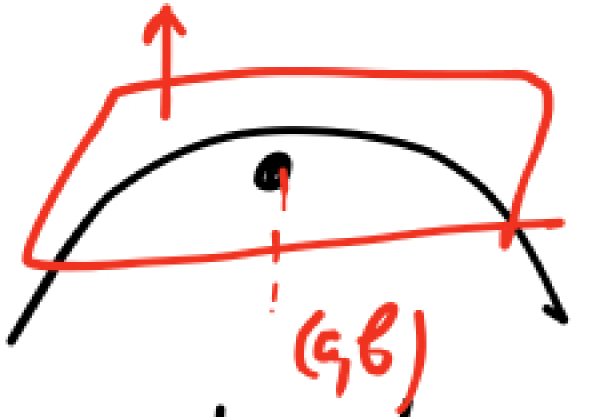
- - - - - **global minimum value**

- - - $f(a,b) \le f(x,y)$ for (x,y) in the domain of f.

How to find local extrema for $f(x,y)$?

Idea

- Assume f is differentiable



$z = f(x,y)$

Tangent plane at local extremum is horizontal

$\Leftrightarrow$ normal vector is parallel to $(0,0,1)$

Recall, normal to tangent plane

$$\vec{n} = (-f_x, -f_y, 1).$$

Then $(-f_x, -f_y, 1) = C\,(0,0,1)$.

$$\Leftrightarrow \quad \left\{ \begin{array}{l} f_x = 0 \\ f_y = 0 \end{array} \right\} \Leftrightarrow \nabla f_{(a,b)} = \vec{0}$$

<u>Upshot</u>: if f has local extremum at (a,b),

then $\nabla f(a,b) = \vec{0}$.