From last time:

- $F(x,y,z) = e^{xz} + xz + z - y - 1 = 0$

near ~~at~~ $(0,0,0)$ defines z = function of x,y.

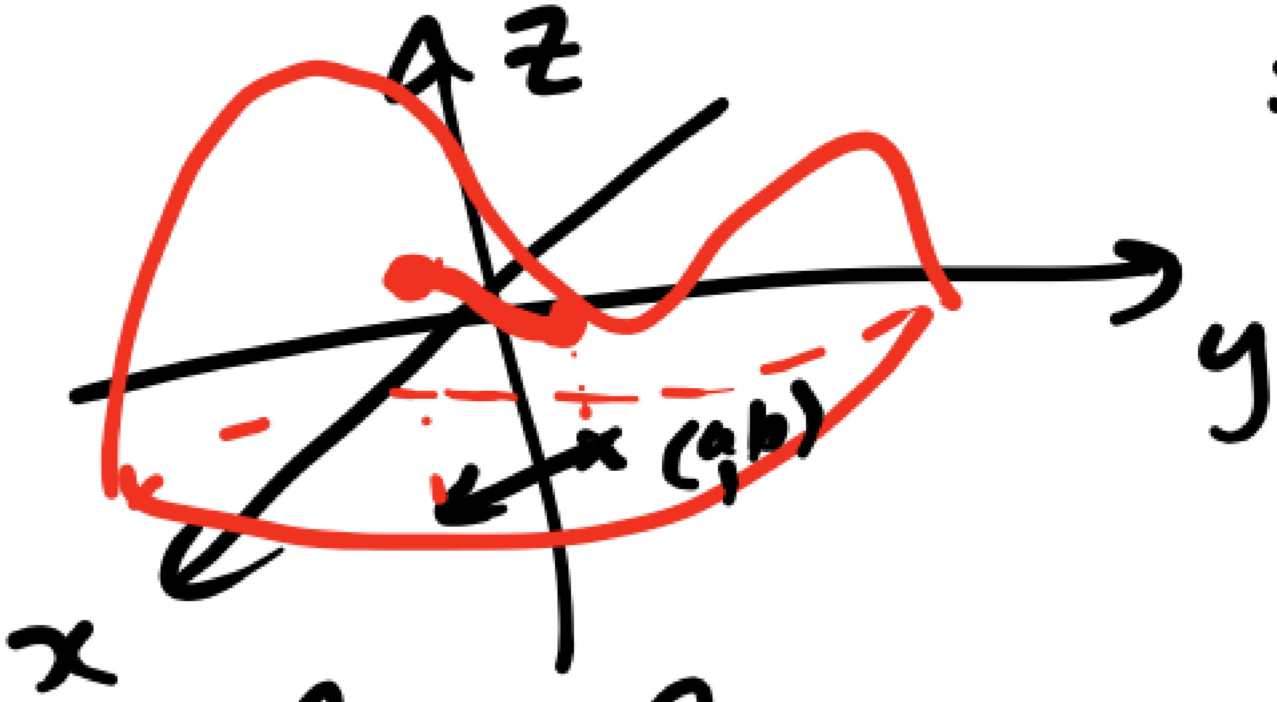
$\longrightarrow$ $F_z = xe^{xz} + x + 1$

$F_z(0,0,0) = 1 \neq 0 \Rightarrow$ By Implicit Function theorem, z is a function of x,y near $(0,0)$

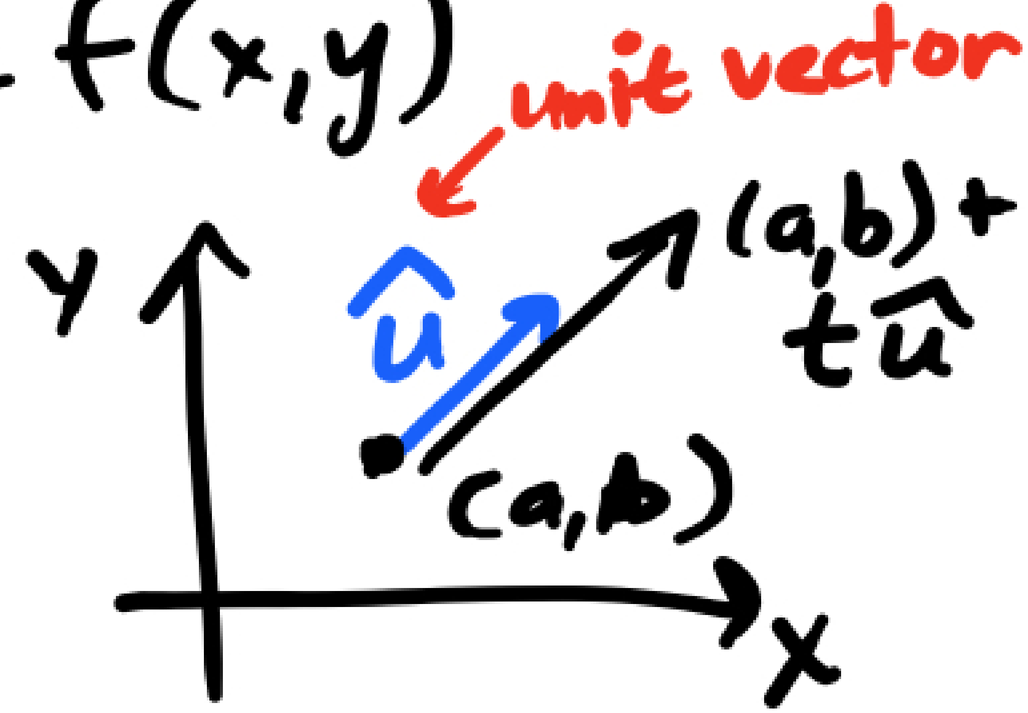
$$\frac{dz}{dx}(0,0) = -\frac{F_x}{F_z}(0,0,0)$$

$$= \ldots = 0, \quad \frac{dz}{dy}(0,0) = 1.$$

14.6. Directional Derivatives.



$z = f(x,y)$



Avr. Rate of change as

$(a,b) \longrightarrow (a,b) + t\hat{u}$ is:

$$\frac{\text{change in } f}{\|\text{change in } (a,b)\|} =$$

$$= \frac{f((a,b)+t\hat{u}) - f(a,b)}{\|\cancel{(a,b)}+t\hat{u} - \cancel{(a,b)}\|} =$$

$$= \frac{f((a,b)+t\hat{u}) - f(a,b)}{t}$$

The instantaneous rate of change of $f(x,y)$ at (a,b) in the direction of the unit vector $\hat{u}$ is:

$$\boxed{\lim_{t \to 0} \frac{f((a,b)+t\hat{u}) - f(a,b)}{t}} = D_{\hat{u}} f(a,b)$$

directional derivative of f at (a,b) in the direction of $\hat{u}$

Example

$\hat{u} = \hat{\imath}$

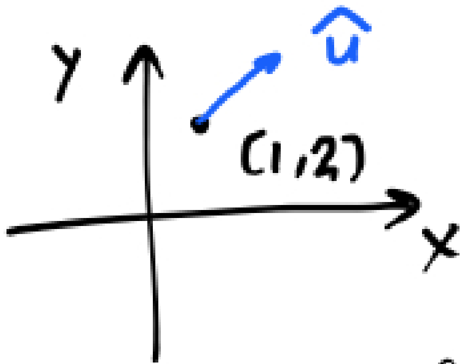
then $\boxed{D_{\hat{\imath}} f = f_x}$

Check: $D_{\hat{\imath}} f(a,b) = \lim\limits_{t \to 0} \dfrac{f((a,b) + t\overset{(1,0)}{\hat{\imath}}) - f(a,b)}{t} =$

$\lim\limits_{t \to 0} \dfrac{f(a+t, b) - f(a,b)}{t} = f_x(a,b).$

Similarly, $\boxed{D_{\hat{\jmath}} f = f_y}$.

<u>Example</u> $f(x,y) = xy$, find $D_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)} f(1,2)$



$$D_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)} f(1,2) =$$

$$= \lim_{t\to 0} \frac{f((1,2)+t\hat{u}) - f(1,2)}{t} =$$

$$= \lim_{t\to 0} \frac{f\left(1+\frac{t}{\sqrt{2}}, 2+\frac{t}{\sqrt{2}}\right) - f(1,2)}{t} =$$

$$= \lim_{t\to 0} \frac{\left(1+\frac{t}{\sqrt{2}}\right)\left(2+\frac{t}{\sqrt{2}}\right) - 1\cdot 2}{t} =$$

$$= \lim_{t\to 0} \frac{\cancel{1\cdot 2} + (1+2)\frac{t}{\sqrt{2}} + \frac{t^2}{2} - \cancel{1\cdot 2}}{t} =$$

$$= \lim_{t\to 0} \left(\frac{3}{\sqrt{2}} + \frac{t}{2}\right) = \boxed{\frac{3}{\sqrt{2}}}$$

→ Is there an easier way to compute directional derivatives?

Yes! Assume f is differentiable at (a,b):

$$f(a+\Delta x, b+\Delta y) \approx \underbrace{f(a,b) + f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y}_{\text{linearization}}$$

Let $\hat{u} = (u_1, u_2)$. Then

$$\frac{f((a,b)+t\hat{u}) - f(a,b)}{t} = \frac{f(a+tu_1, b+tu_2) - f(a,b)}{t}$$

$$\underset{(\text{if } t\approx 0)}{\approx} \frac{\cancel{f(a,b)} + f_x(a,b)\,tu_1 + f_y(a,b)\,tu_2 - \cancel{f(a,b)}}{t} =$$

$$= \underline{f_x(a,b)\cdot u_1 + f_y(a,b)\cdot u_2} \leftarrow D_{\hat{u}} f(a,b)$$

$$= \left(f_x(a,b), f_y(a,b)\right) \cdot (\underbrace{u_1, u_2}_{\hat{u}}) =$$

$$= \boxed{\nabla f(a,b) \cdot \hat{u}}\,.$$

Def For $f(x,y)$ its <u>gradient vector</u> at point (a,b) is the vector

$$\nabla f(x,y) = \left(f_x(a,b), f_y(a,b)\right)$$

[another notation: grad f].

Example $f(x,y) = xy$, find $D_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})} f(1,2)$.

$$\nabla f(x,y) = (y, x)$$

$$D_{\vec{u}} f(1,2) = (2,1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) =$$

$$2 \cdot \frac{1}{\sqrt{2}} + 1 \cdot \frac{1}{\sqrt{2}} = \boxed{\frac{3}{\sqrt{2}}}$$

Note Same works for functions of more variables

$$\nabla f(x_1, \dots, x_n) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right).$$

and then

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}.$$