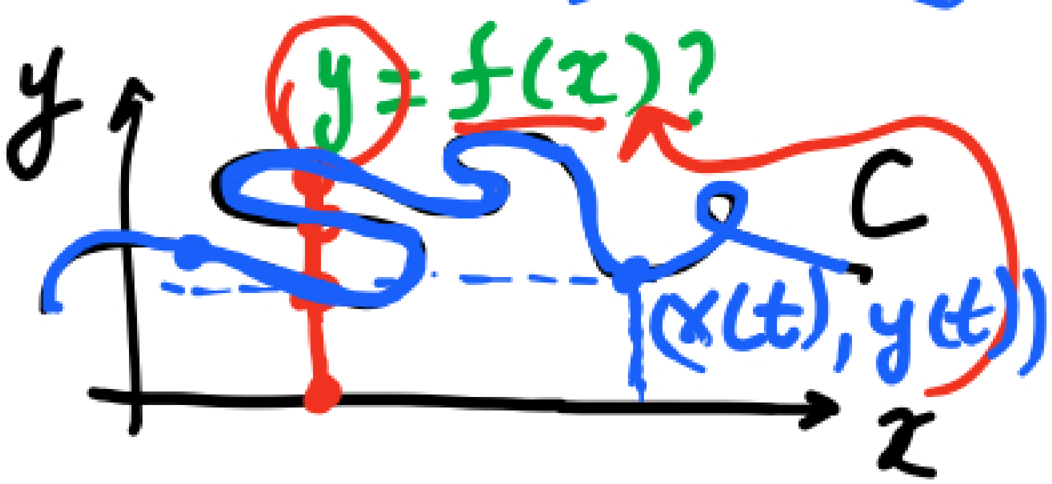
10.1.

Curves defined by Parametric Equations



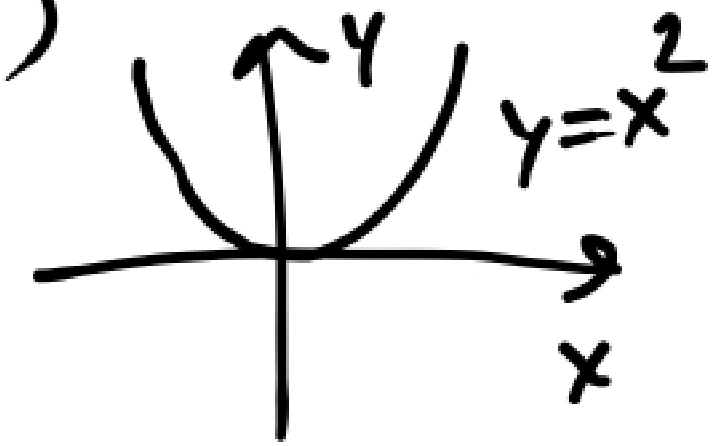
C is not a graph of function $y = f(x)$
but a curve on the plane

Q: How to work with curves?

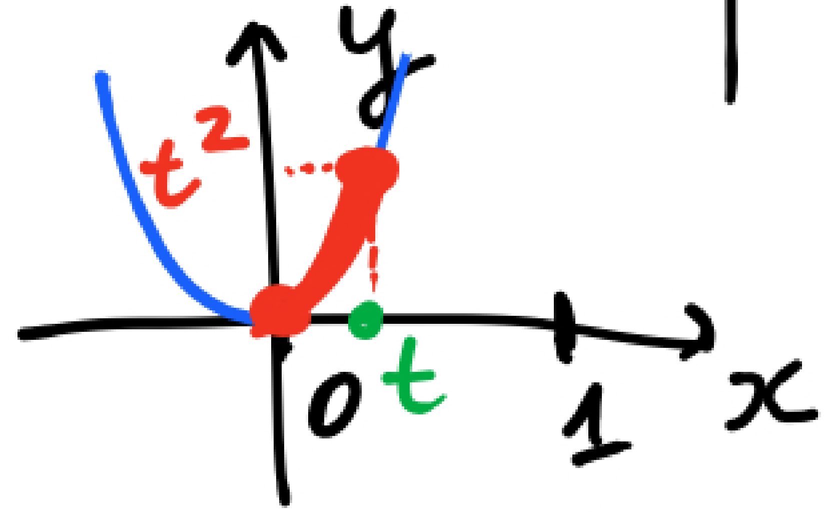
A: Think of drawing the curve with a pencil and a clock. Then at time t you reach the point $x(t), y(t)$ on the plane

Parametric curve: $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

Examples ① $y = f(x)$

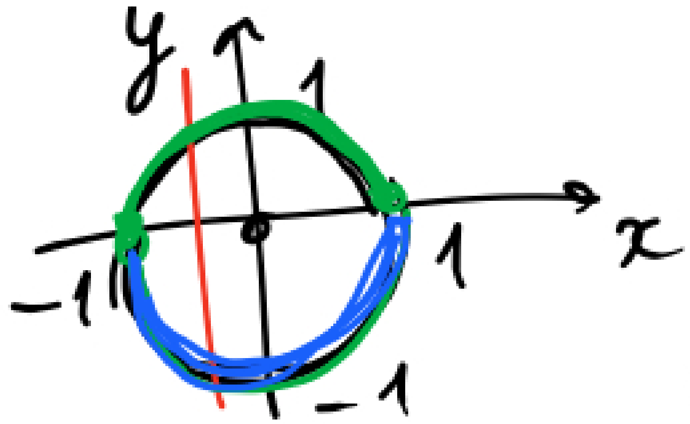


we can take $\begin{cases} x = t \\ y = t^2 \end{cases}$



$0 \leq t \leq 1.$

② circle

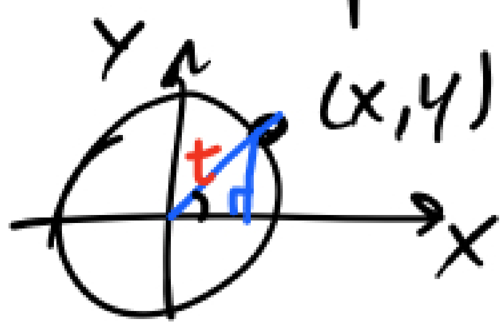


$$x^2+y^2=1.$$

$$\hookrightarrow\; y=\pm\sqrt{1-x^2}$$

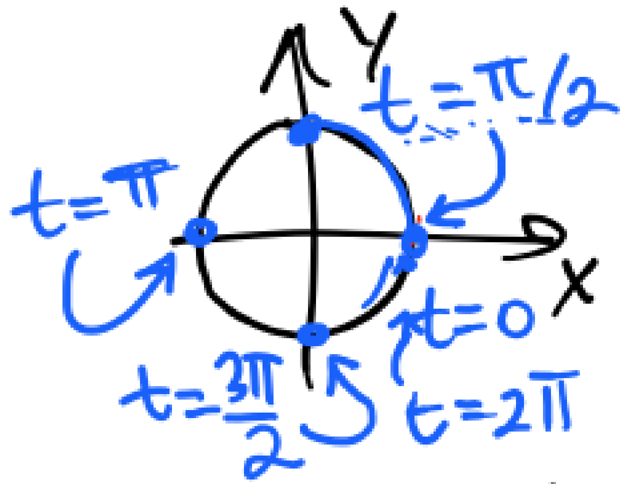
$$\begin{cases} x = t \\ y = \pm\sqrt{1-t^2} \end{cases} \quad -1 \le t \le 1$$

a better parametrization:



$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

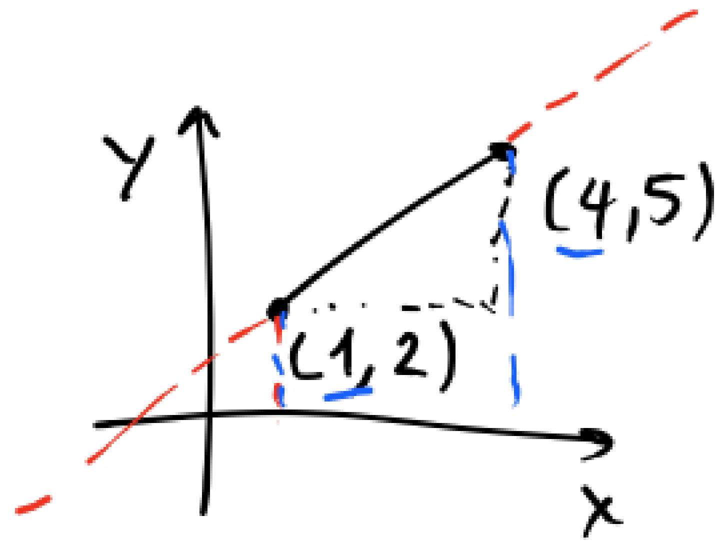
$$0 \le t \le 2\pi$$



If radius is r : $\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$

Upshot: one curve admits many parametrizations.

③ Lines



Equation:

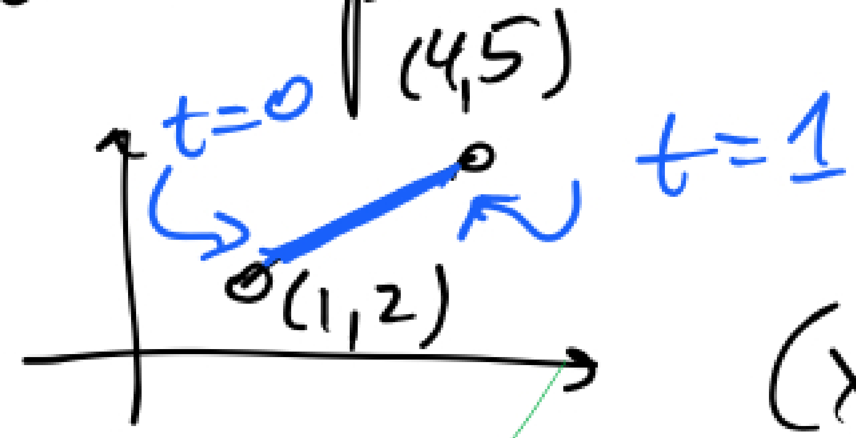
$$\text{slope} = \frac{5-2}{4-1} = 1.$$

Then $y-2 = 1\cdot(x-1)$

$y = x+1.$

$$\boxed{\begin{cases} x = t \\ y = t+1 \quad 1 \le t \le 4 \end{cases}}$$

another parametrization:



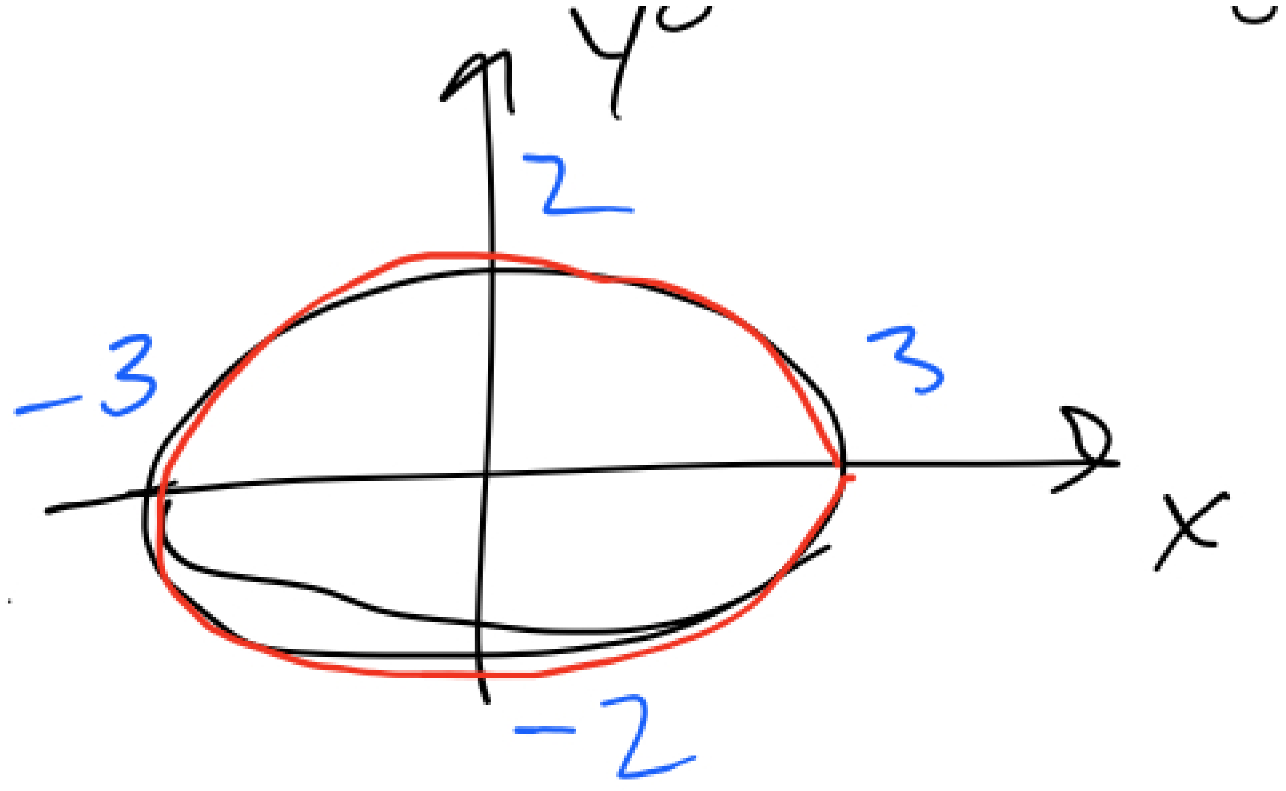
$$(x(t), y(t)) =$$

$$= \underline{(1-t)}\,(1,2) + \underline{t}\,(4,5) =$$

$$= (\underbrace{1+3t}_{x(t)}, \underbrace{2+3t}_{y(t)}) \qquad 0 \le t \le 1$$

- works for every line segment.

(4) ellipse



$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$\underbrace{\left(\frac{x}{3}\right)^2}_{\cos t} + \underbrace{\left(\frac{y}{2}\right)^2}_{\sin t} = 1$$

$$\begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases}$$

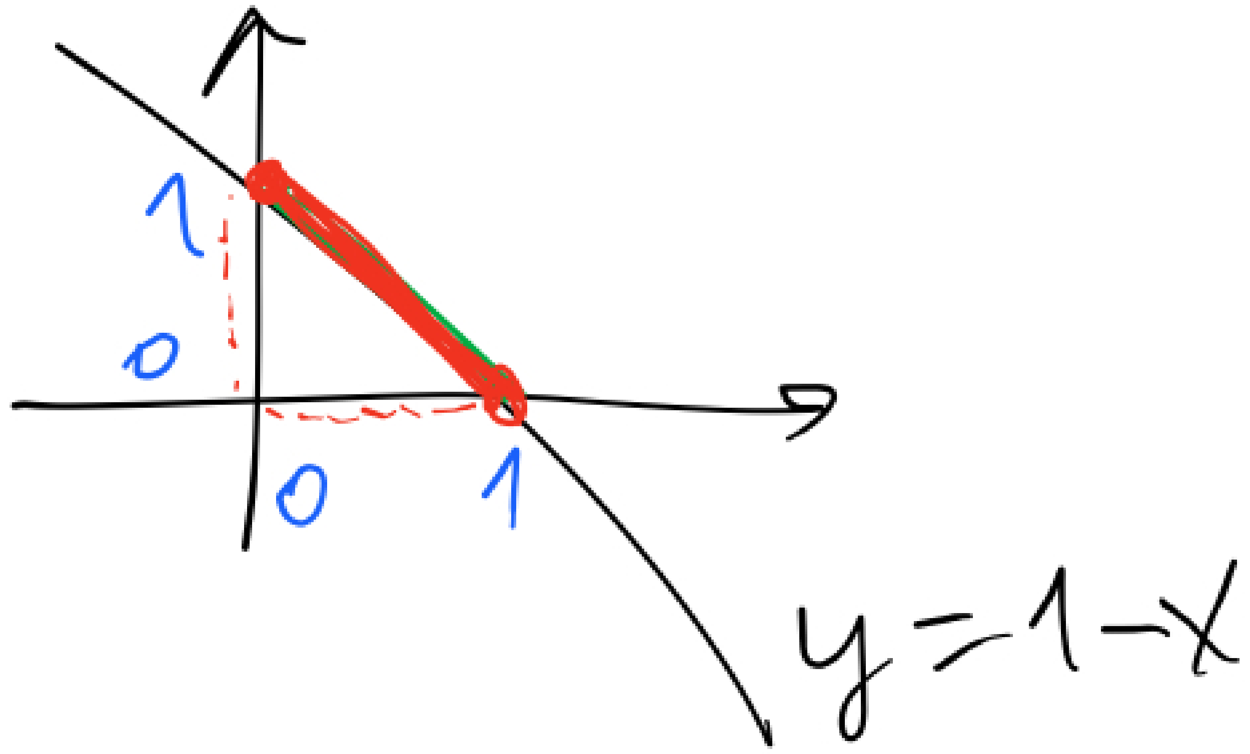
$$0 \le t \le 2\pi$$

(5) $\begin{cases} x = \cos^2 t \\ y = \sin^2 t \end{cases}$ what parametric curve do we get?

$\Downarrow$

$$x + y = \cos^2 t + \sin^2 t = 1.$$

$$y = 1 - x.$$



$-1 \le \sin t \le 1$

$-1 \le \cos t \le 1$

$0 \le \sin^2 t \le 1$

$0 \le \cos^2 t \le 1$