Applications of the Chain Rule.

Example $w = xyz^2$ $\quad x = r+s+t$

$y = rs$

$z = se^t$

$\frac{\partial w}{\partial r}, \frac{\partial w}{\partial s}, \frac{\partial w}{\partial t} = ?$

$$\frac{\partial w}{\partial r} = \underbrace{\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}} =$$

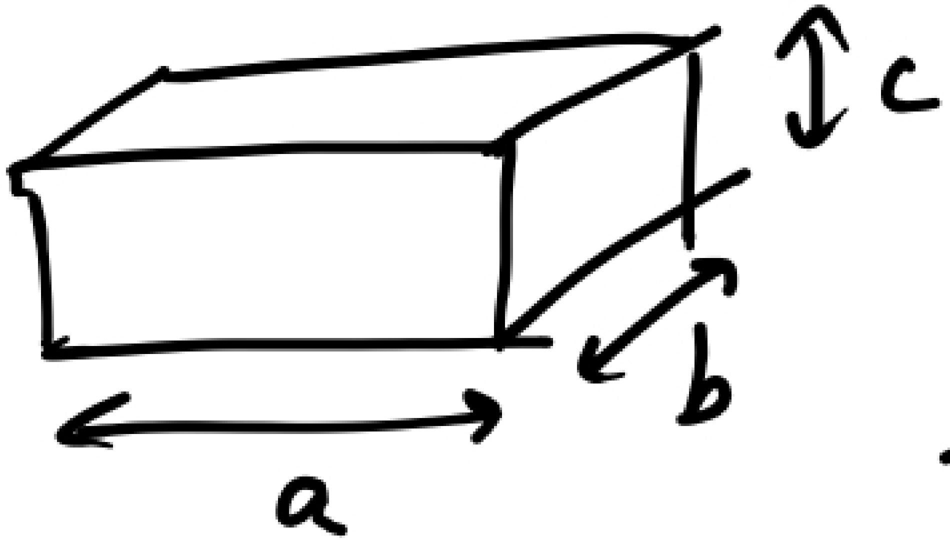
$$= yz^2 \cdot 1 + xz^2 \cdot s + 2xyz \cdot 0 =$$

$$= rs\,(se^t)^2 + (r+s+t)(se^t)^2 \cdot s.$$

etc.

I. Related Rates

Suppose that a rectangular ice box is melting at a rate so it's sides are decreasing 1, 2 and 3 cm/s. How fast is the volume changing when the sides are 10, 10, 20 cm?



$V = abc$ and $\begin{aligned} a &= a(t) \\ b &= b(t) \\ c &= c(t) \end{aligned}$ $\quad \left.\dfrac{dV}{dt}\right|_{\substack{(a,b,c)\\=(10,10,20)}} = ?$

Chain Rule: $\left.\dfrac{dV}{dt}\right|_{(a,b,c)=(10,10,20)} = \dfrac{dV}{da}\cdot\dfrac{da}{dt} + \dfrac{dV}{db}\cdot\dfrac{db}{dt}$

$$+\ \frac{dV}{dc}\cdot\frac{dc}{dt} =$$

$$= bc\cdot(-1) + ac(-2) + ab(-3) =$$

$$= 200(-1) + 200(-2) + 100(-3) =$$

$$= -900\ cm^3/s.$$

II. Implicit differentiation.

Suppose $F(x,y)=0$ implicitly defines y as a function of x

e.g. • $y-x^2=0 \longrightarrow y=x^2$

• $y^2+x^2-1=0 \longrightarrow y=\pm\sqrt{1-x^2}$

• $y\sin y+x^2=0 \longrightarrow y=\,?$

What is $\frac{dy}{dx}=\,?$

Let's use Chain Rule: $F(x,y) = 0$

$$\rightsquigarrow \frac{d}{dx}\left[F(x,y)\right] = \frac{d}{dx}[0]$$

$$\cancel{F_x} = 0.$$

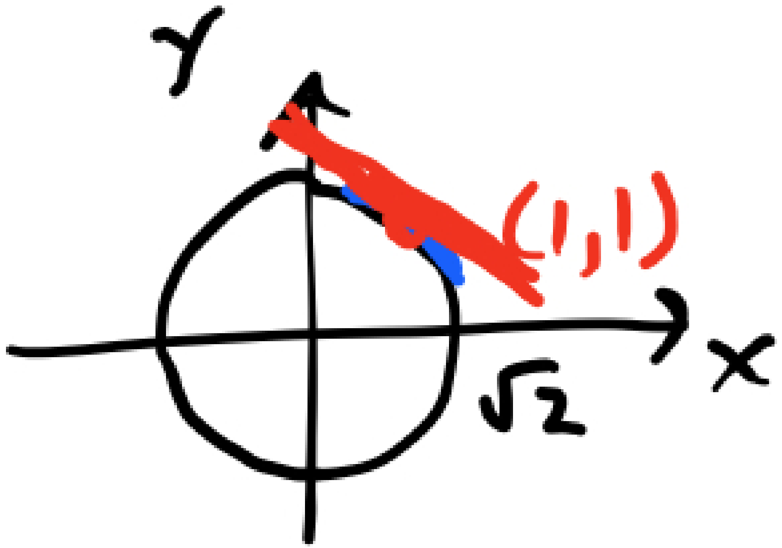
$$F(x,y) = F(x, y(x))$$

$$\frac{d}{dx}\left[F(x,y(x))\right] = F_x \underbrace{\frac{dx}{dx}}_{=1} + F_y \frac{dy}{dx}$$

$$\Rightarrow F_x + F_y \frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y} \quad (\text{if } F_y \neq 0)}$$

Example $\underbrace{x^2+y^2-2}_{F(x,y)}=0$



$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{2y}$$

$$\frac{dy}{dx}(1,1) = -\frac{1}{1} = -1. \checkmark$$

Suppose $F(x,y,z)=0$ defines z as a function of x and y.

e.g.

- $z-x^2-y^2=0 \longrightarrow z=x^2+y^2$
- $e^{xz}+xz+z-y-1=0 \longrightarrow z=\,??$

What are $\frac{dz}{dx}$ and $\frac{dz}{dy}$?

Chain Rule:

$$\frac{d}{dx}\left(F(x,y,z(x,y))\right) =$$

$$= F_x \cdot \underbrace{\frac{dx}{dx}}_{=1} + F_y \cdot \underbrace{\frac{dy}{dx}}_{=0} + F_z \cdot \frac{dz}{dx} = 0$$

x and y are independent

$$\Rightarrow \boxed{\frac{dz}{dx} = -\frac{F_x}{F_z} \quad (\text{if } F_z \neq 0)}$$

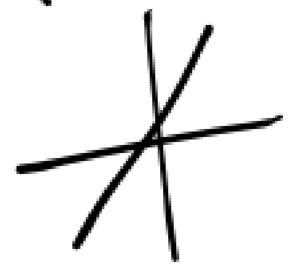
Similarly,

$$\boxed{\frac{dz}{dy} = -\frac{F_y}{F_z} \quad (\text{if } F_z \neq 0)}$$

When does it make sense?

[z defines a function of x and y]

Implicit function theorem



If $F(x,y,z)=0$ and

$F_z(a,b,c) \neq 0$ and F_x, F_y, F_z are continuous,

then $z =$ function of x and y near (a,b)

Moreover, it is differentiable with

$$\frac{dz}{dx} = -\frac{F_x}{F_z}.$$

$$\frac{dz}{dy} = -\frac{F_y}{F_z}.$$

Example

Exercise

$$F(x,y,z) = \underbrace{e^{xz} + xz + z - y - 1 = 0}$$

at $\underset{F=0}{\underline{(0,0,0)}}$

$$F_z = \ldots$$