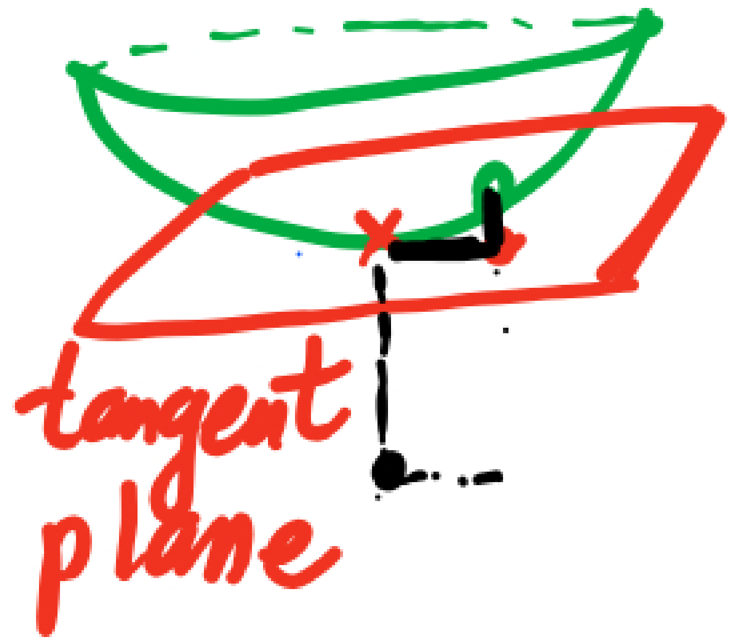
Last time:

- differentiability

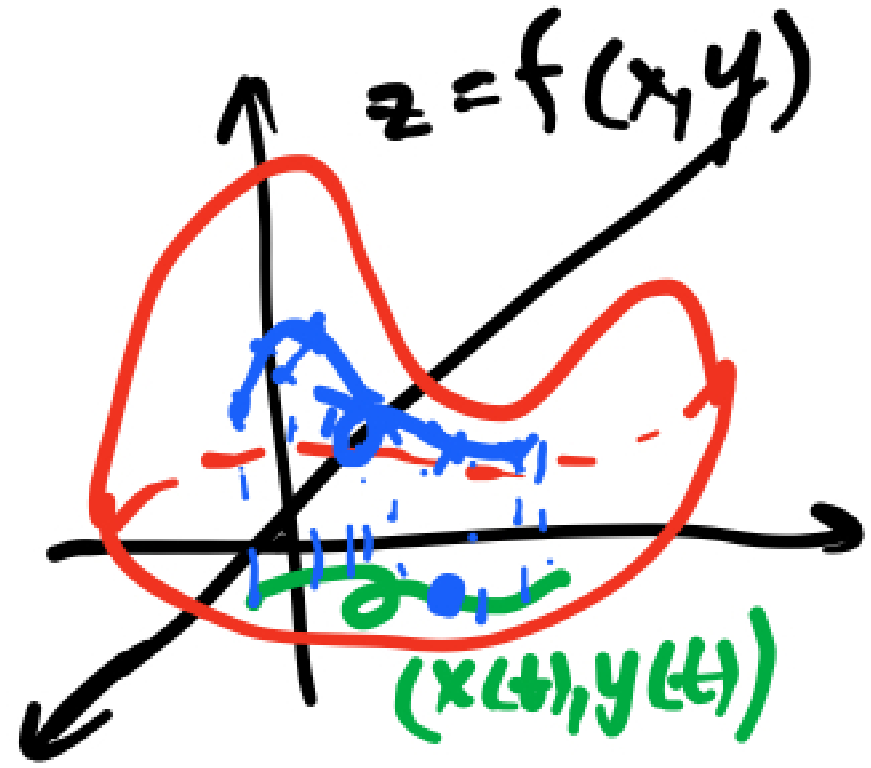


The Chain Rule.

Question: If $z = f(x, y)$ and

$$\begin{cases} x = \underline{x(t)} \\ y = \underline{y(t)} \end{cases}$$

What is $\dfrac{dz}{dt}$?



Example $\quad z = xy$ and

$$\begin{cases} x = 1+t^2 \\ y = e^t \end{cases}$$

$$\rightsquigarrow \quad z = (1+t^2)e^t$$

$$\frac{dz}{dt} = 2t \cdot e^t + (1+t^2)e^t.$$

Instead : Calc 1 $\quad y = f(x)$,

$$x = g(t)$$

$$\rightsquigarrow \quad y = f(g(t)).$$

$$y = e^x$$

$$x = t^2+1$$

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

Chain Rule.

$$\frac{dy}{dt} = e^{t^2+1} \cdot (2t)$$

If $z = f(x,y)$ is differentiable,
(and $x(t), y(t)$ too).

then

$$\boxed{\frac{dz}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}}$$

Chain Rule

can be shown by using linear approximation of $f(x,y)$

$z \approx f_x(x(t)-a) + f_y(y(t)-b) + f(a,b)$

$z(t+h$

Verify: $z = xy$ and $\begin{cases} x = 1+t^2 \\ y = e^t \end{cases} \rightsquigarrow \frac{dz}{dt} = e^t \cdot 2t + (1+t^2)e^t.$

__Question:__ If $z=f(x,y)$ and $\begin{array}{l} x=x(u,v) \\ y=y(u,v) \end{array}$, then

z becomes a function of u and v.

What are $\dfrac{dz}{du}$ and $\dfrac{dz}{dv}$? "Same"

$$\boxed{\frac{dz}{du}=f_x\frac{dx}{du}+f_y\frac{dy}{du}} \qquad \boxed{\frac{dz}{dv}=f_x\frac{dx}{dv}+f_y\frac{dy}{dv}}$$

$$\frac{dz}{du}=\frac{dz}{dx}\cdot\frac{dx}{du}+\frac{dz}{dy}\cdot\frac{dy}{du} \qquad \frac{dz}{dv}=\frac{dz}{dx}\cdot\frac{dx}{dv}+\frac{dz}{dy}\cdot\frac{dy}{dv}$$

Example $z = x^2 + y^2$ $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta}$?

$$\frac{dz}{dr} = \frac{dz}{dx} \cdot \frac{dx}{dr} + \frac{dz}{dy} \cdot \frac{dy}{dr} =$$

$$2x \frac{dx}{dr} + 2y \frac{dy}{dr} =$$

$$2x (\cos\theta) + 2y (\sin\theta) =$$

$$= 2r\cos^2\theta + 2r\sin^2\theta = \underline{2r}.$$

$$\frac{dz}{d\theta} = \frac{dz}{dx} \cdot \frac{dx}{d\theta} + \frac{dz}{dy} \cdot \frac{dy}{d\theta} =$$

$$2x \cdot \frac{dx}{d\theta} + 2y \frac{dy}{d\theta} =$$

$$= 2x(-r\sin\theta) + 2y(r\cos\theta) =$$

$$= -2r^2 \sin\theta\cos\theta + 2r^2 \sin\theta\cos\theta =$$

$$= \underline{0}.$$

<u>Check</u>: $z = x^2 + y^2 = r^2$

$$\frac{dz}{dr} = 2r, \quad \frac{dz}{d\theta} = 0.$$

General form of the Chain Rule:

If $z = f(x_1, \dots, x_n)$ and

$$x_1 = x_1(t_1, \dots, t_m)$$
$$x_2 = x_2(t_1, \dots, t_m)$$
$$\vdots$$
$$x_n = x_n(t_1 \dots t_m)$$

Then

$$\boxed{\begin{aligned} \frac{dz}{dt_i} = \frac{dz}{dx_1}\cdot\frac{dx_1}{dt_i} + \frac{dz}{dx_2}\cdot\frac{dx_2}{dt_i} + \dots \\ + \frac{dz}{dx_n}\cdot\frac{dx_n}{dt_i} \end{aligned}}$$

$i = 1 \dots m$

or $$\boxed{\frac{dz}{dt_i} = \sum_{j=1}^{n} \frac{dz}{dx_j} \cdot \frac{dx_j}{dt_i}}$$

Ex: $w = xyz^2$

$$x = r+s+t$$
$$y = r \cdot s$$
$$z = se^t$$

$\frac{dw}{dr}, \frac{dw}{ds}, \frac{dw}{dt}$?