Def. The best linear approximation (or linearization) of $f(x,y)$ near (a,b) is

$$L = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

e.g. for $f(x,y) = 3 - x^2 - y^2$ near $(1,1)$

$$f(1.01, 1.02) = 0.9395 \text{ and}$$

$$L(1.01, 1.02) = 0.94.$$

Differentiability

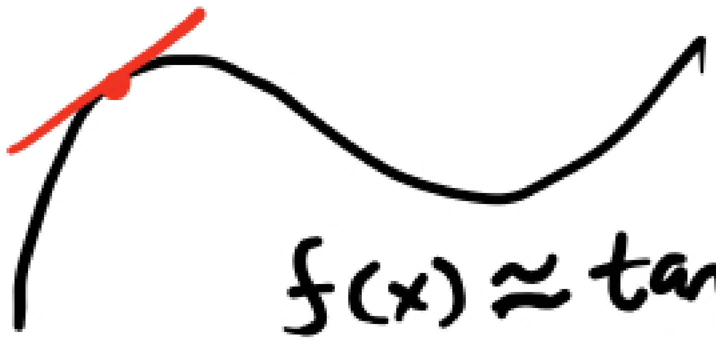
- $f(x)$ is differentiable at $x=a$ if $f'(a)$ exists.

- $f(x,y)$ is differentiable at $(x,y)=(a,b)$

  if ... ?

→ ~~if $f_x(a,b)$ and $f_y(a,b)$ exist~~ ?

not enough! doesn't even guarantee continuity.



$f(x) \simeq$ tangent line
$x \simeq a$

$f(x,y)$ is differentiable if:
$f(x,y) \approx$ tangent plane
$(x,y) \approx (a,b)$

Formally:

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - L(x,y)|}{\|(x,y)-(a,b)\|} = 0$$

$$\lim_{x\to a} \frac{f(x) - [f(a)+f'(a)\cancel{(x-a)}]}{\cancel{x-a}} = 0$$

e.g.

$$\lim_{(x,y)\to(1,1)} \frac{3-x^2-y^2-(5-2x-2y)}{\|(x,y)-(1,1)\|} = 0$$

$$\lim_{(x,y)\to(1,1)} \frac{3-x^2-y^2-5+2x+2y}{\sqrt{(x-1)^2+(y-1)^2}} = 0$$

Hard to check in practice :'(

Theorem If $f(x,y)$ has partial derivatives f_x and f_y near (a,b) that are continuous at (a,b), then $f(x,y)$ is differentiable at (a,b).

(cts partials $\Rightarrow$ differentiability)

* Converse not true

not differentiable!

Example $f(x,y) = 3 - x^2 - y^2$

$$\left.\begin{aligned} f_x &= -2x \\ f_y &= -2y \end{aligned}\right\} \text{ continuous everywhere} \Rightarrow f \text{ is differentiable everywhere.}$$

14.5. Chain Rule

Question If $z = f(x, y)$ and $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$,

then z becomes a function of t.

What is $\frac{dz}{dt}$?

Example $z = xy$ and $\begin{cases} x = 1+t^2 \\ y = e^t \end{cases}$

$$\rightsquigarrow \quad z = (1+t^2)e^t$$

$$\frac{dz}{dt} = 2t \cdot e^t + (1+t^2)e^t.$$

Instedd: Calc 1 $\quad y = f(x)\,,\; x = g(t)$

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$