Partial derivatives

(reminder)

Def^n The partial derivative of $f(x,y)$ at (a,b) with respect to x is given by

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

(Similarly w.r.t. y).

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

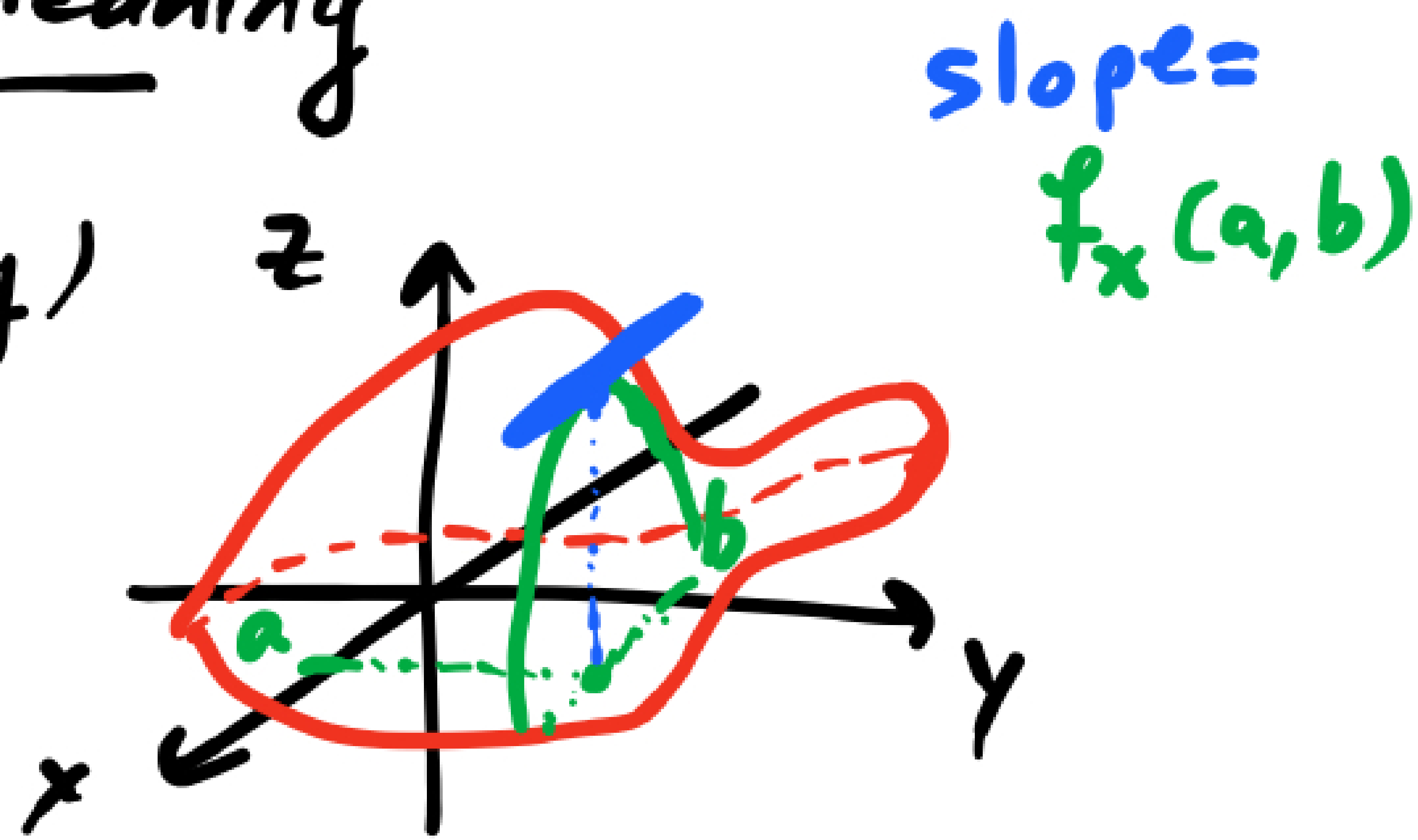
Observation To compute f_x treat y as a constant and differentiate wrt x as if it was a single var. function

E.g. $f(x,y) = x^2 y$

$f_x = 2xy \qquad f_y = x^2$

Geometric meaning

$z = f(x,y)$



Notation:

$z = f(x,y)$

p.d. wrt x: f_x, $\dfrac{\partial f}{\partial x}$, $\dfrac{df}{dx}$, z_x, $\dfrac{\partial z}{\partial x}$, ...

partial der.

What if there are more variables?

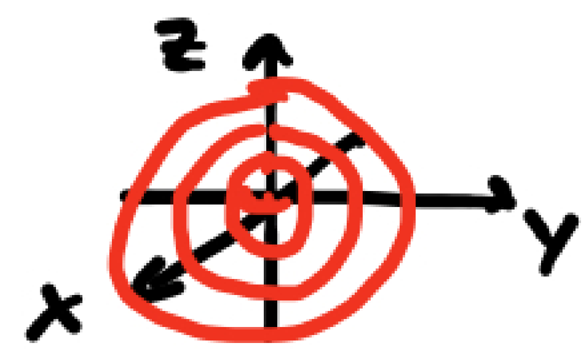
$w = f(x_1, x_2, \ldots, x_n)$ is a "hypersurface" in $\mathbb{R}^{n+1}$

e.g. $w = x^2+y^2+z^2$

Partial derivatives still make sense: $f_{x_1}, \ldots, f_{x_n}$.

Level surfaces: $w = k$

$k = x^2+y^2+z^2$



compare:

$z = x^2+y^2$

Example $w = \ln\left(\frac{x}{y}\right) - xe^{xyz}$.

Find w_x, w_y, w_z.

$$w = \ln x - \ln y - x \cdot e^{xyz}.$$

$$w_x = 1/x - 0 - (\underbrace{1 \cdot e^{xyz} + x \cdot yze^{xyz}}_{\text{product rule}})$$

$$w_y = 0 - 1/y - x(xze^{xyz})$$

$$w_z = 0 - 0 - x(xye^{xyz}).$$

Higher order partial derivatives.

$$
\begin{tikzcd}
f(x,y) \arrow[r, "\text{diff}", "\text{wrt } x"'] \arrow[d, "\text{wrt } y", "\text{diff}"'] & f_x(x,y) \arrow[r, "\text{diff}", "\text{wrt } x"'] \arrow[d, "\text{wrt } y", "\text{diff}"'] & f_{xx}(x,y) \\
f_y(x,y) \arrow[d, "\text{wrt } y", "\text{diff}"'] \arrow[dr, "\text{diff}", "\text{wrt } x"'] & (f_x)_y(x,y) \arrow[d, phantom, "\overset{?}{=}"] & \\
f_{yy}(x,y) & (f_y)_x(x,y) &
\end{tikzcd}
$$

WARNING

In general, higher order partial derivatives depend on order of diff!

Notation

$f_{xx} = (f_x)_x \qquad f_{yx} = (f_y)_x$

$f_{xy} = (f_x)_y \qquad$ etc.

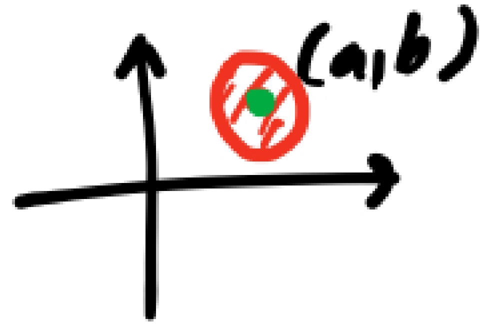
Example $f(x,y) = x^2y + \sin x$

$f_x = 2xy + \cos x \qquad f_y = x^2$

$f_{xy} = 2x \qquad f_{yx} = 2x$

Clairaut's Theorem

(equality of mixed partial derivatives)



If $f_{xy}(x,y)$ and $f_{yx}(x,y)$ exist near (a,b) and continuous at (a,b)

then $f_{xy}(a,b) = f_{yx}(a,b)$