# 14.3. Partial derivatives

Motivation: $f(x,y)$

how does it change when $x$ and $y$ change?

Recall, the rate of change of $f(x)$ is controlled by derivative.

Question: How to define "derivative" for a multivar. function?

## Idea behind partial derivatives:

hold one variable fixed and allow the other to vary.

$$f(x,y)$$

at point $(x_0, y_0)$

function of $x$ $\quad f(x, y_0) \qquad\qquad f(x_0, y)$ function of $y$

derivative

partial derivative w.r.t. $x$ $\quad \dfrac{\partial f}{\partial x}(x_0, y_0) \qquad\qquad \dfrac{\partial f}{\partial y}(x_0, y_0)$ p.d. w.r.t. $y$.

__Def$^n$__ The partial derivative of $f(x,y)$ at $(a,b)$ with respect to $x$ is given by

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

another notation

(Similarly w.r.t. $y$)

Examples $f(x,y) = x^2 y$.

Find partial derivative wrt $x$ at $(1,0)$.

$$f_x(1,0) = \lim_{h\to 0} \frac{f(1+h,0) - f(1,0)}{h} =$$

$$= \lim_{h\to 0} \frac{(1+h)^2 \cdot 0 - 1^2 \cdot 0}{h} = \lim_{h\to 0} 0 = 0$$

what about $(x,y)$?

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} =$$

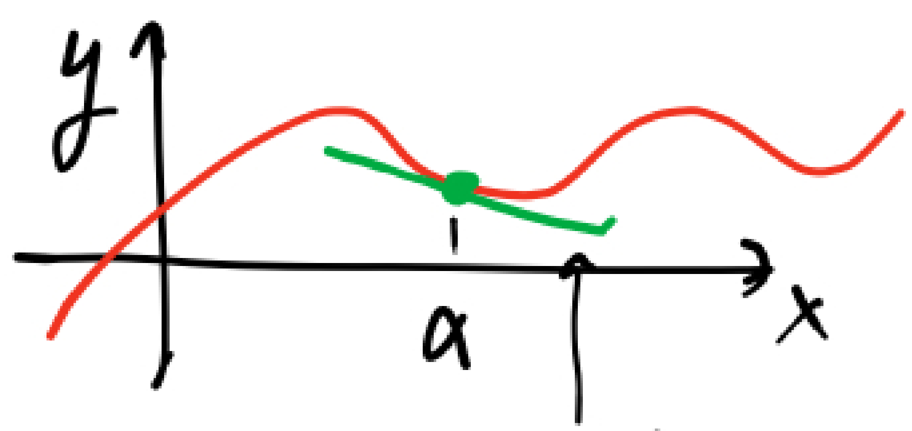
$$= \lim_{h\to 0} \frac{(x+h)^2 y - x^2 y}{h} = \lim_{h\to 0} \frac{(\cancel{x^2} + 2xh + h^2)y - \cancel{x^2 y}}{h}$$

$$= \lim_{h\to 0} \frac{2x\cancel{h}y + h^{\cancel{2}}y}{\cancel{h}} = \lim_{h\to 0} (2xy + hy) = 2xy$$

<u>Observation</u>. To compute $f_x$ treat $y$ as a constant and differentiate wrt $x$ as if it was a single variable function.

## Geometric meaning of partial derivatives

Recall

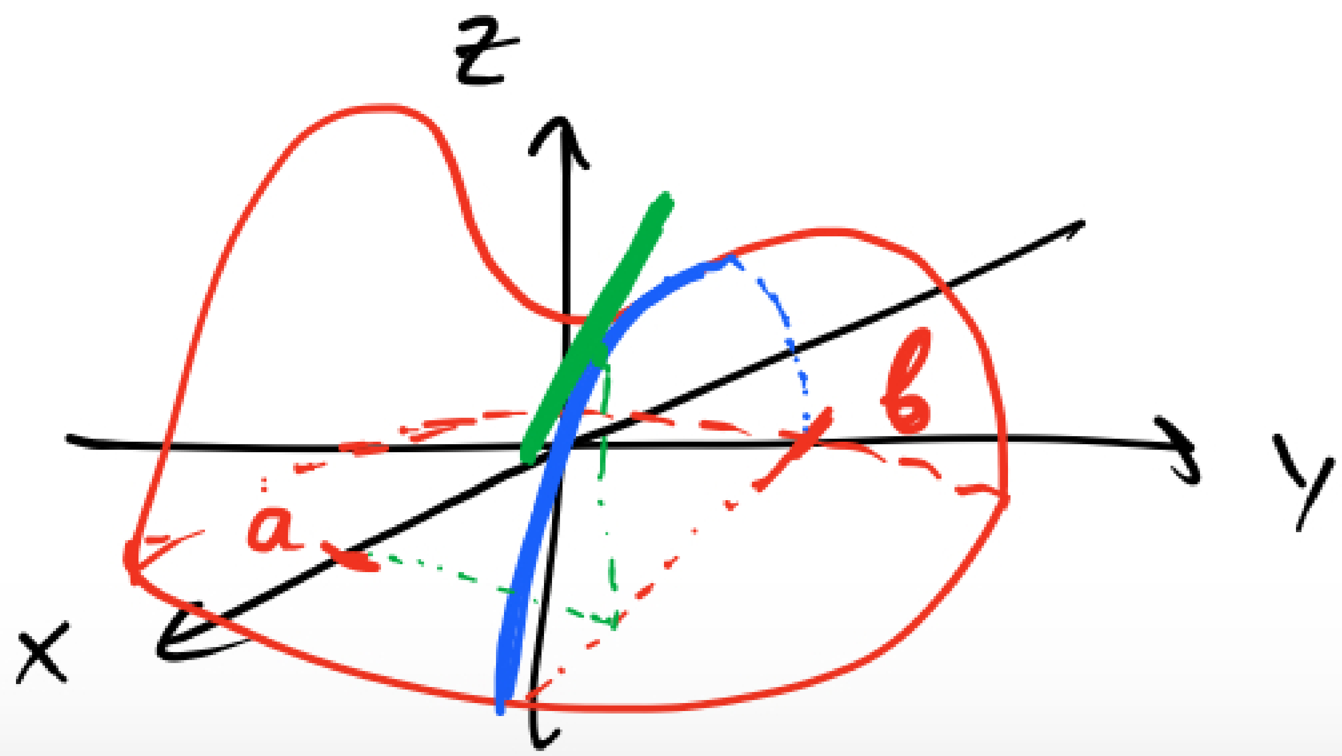



$y = f(x)$

slope $= f'(a)$

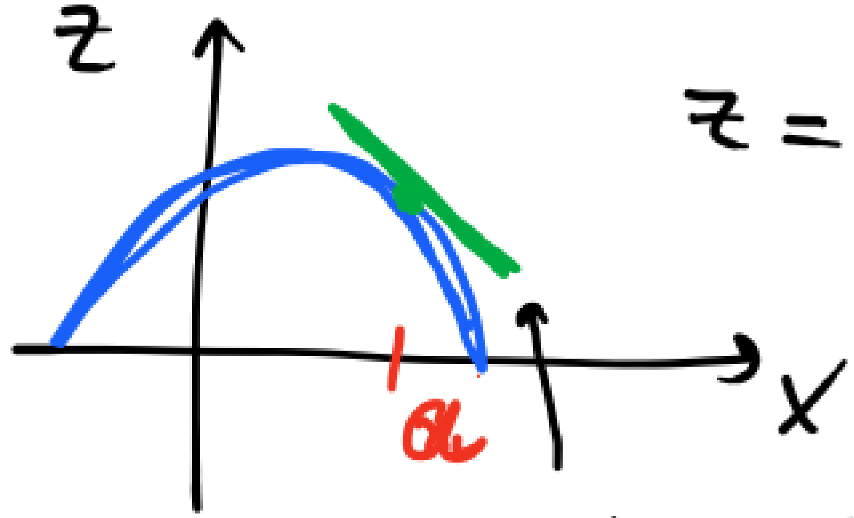
$z = f(x,y)$

Fix $y = b$




$f_x(a,b)$

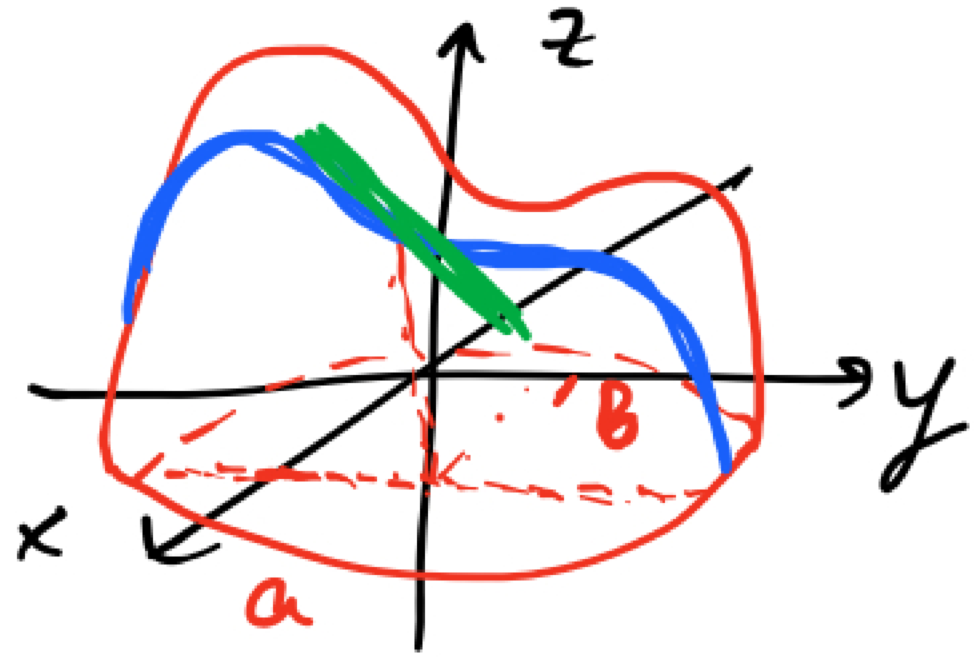
$z = f(x, b)$



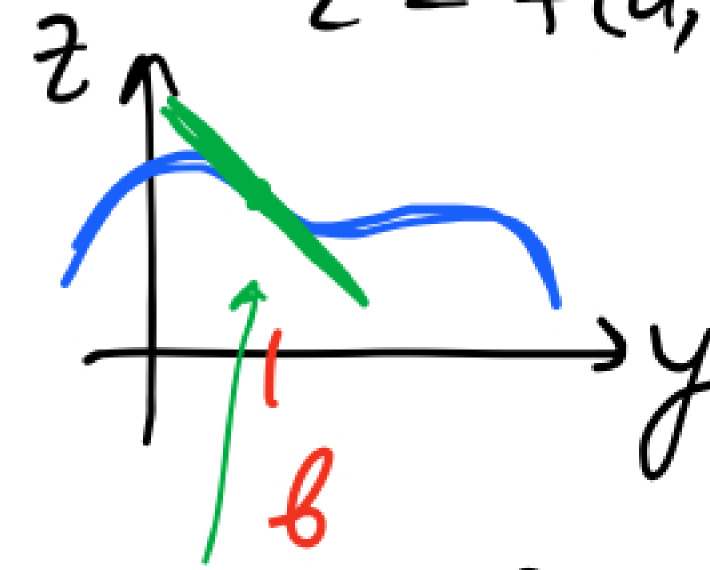

$\text{slope} = f_x(a, b)$

$z = f(x, y)$

Fix $x = a$

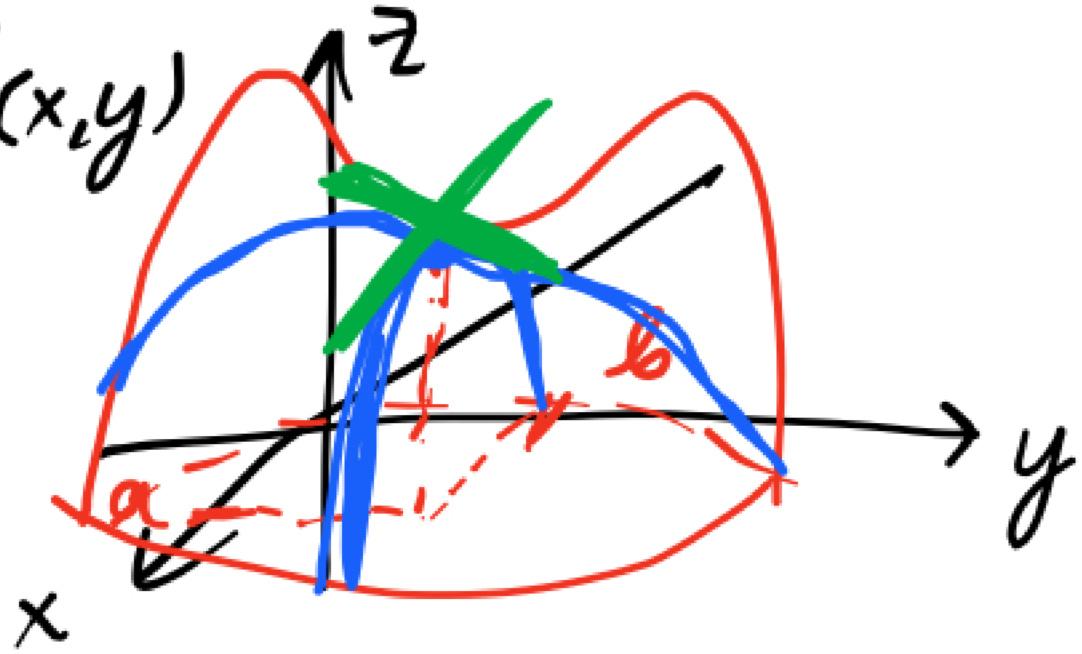



$z = f(a, y)$




$\text{slope} = f_y(a, b)$

$z = f(x, y)$



__Examples__ ① $f(x,y) = \tan(x+2xy^2)$

$f_x = ?$ $\quad f_y = ?$

$f = \tan(\underbrace{(1+2y^2)}_{\text{constant wrt } x}\, x)$ $\quad f_x = (1+2y^2)\sec^2((1+2y^2)x)$

(since $\frac{d}{dx}(\tan(kx)) = k\sec^2 kx$)

$f_y \overset{\text{chain rule}}{=} \sec^2(x+2xy^2)\cdot \underbrace{(x+2xy^2)'}_{\text{wrt } y} =$

$= \sec^2(x+2xy^2)\cdot(0+2x\cdot 2y) =$

$4xy \cdot \sec^2(x+2xy^2)$

② $f(r,\theta) = r\cos\theta$

$f_r = \cos\theta$ $\quad f_\theta = -r\sin\theta$

③ $f(x,y) = xe^{xy}$. Exercise