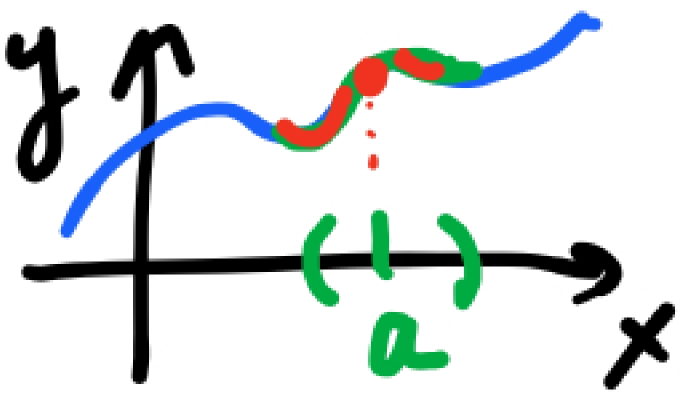
14.2. Limits and continuity

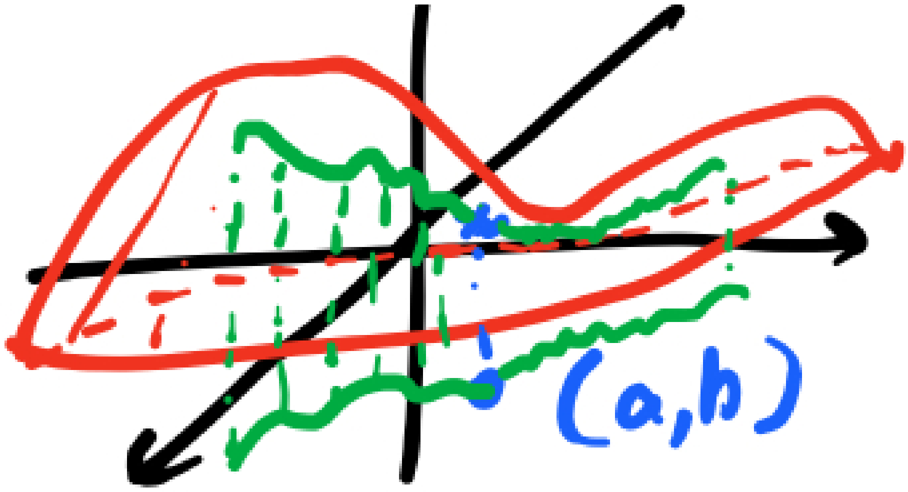


$y = f(x)$

Defn $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

Similarly,

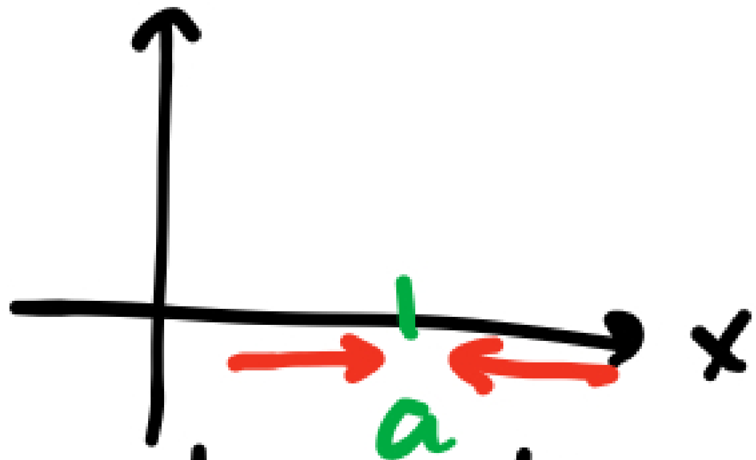
Defn $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.



The notion of "$\lim\limits_{(x,y)\to(a,b)}$" is <u>delicate</u>:

there are infinitely many ways to approach (a,b)!

· In Calc 1:



$\lim\limits_{x\to a^-}$ $\lim\limits_{x\to a^+}$

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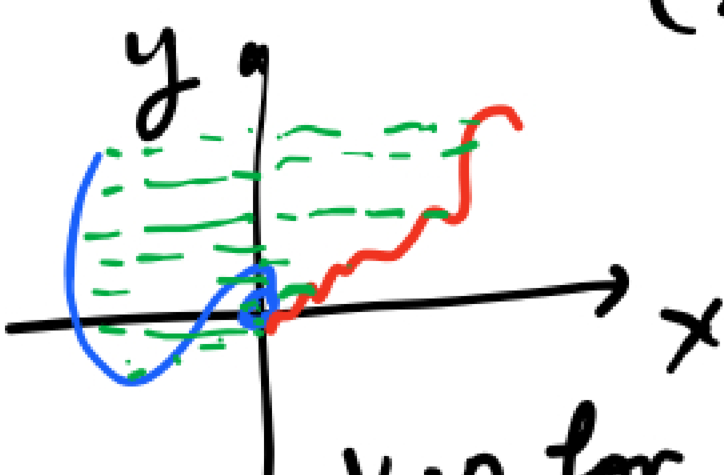
"Intuitive" defn of $\lim_{(x,y)\to(a,b)} f(x,y) = L$:

- Take limit along every path in $\mathbb{R}^2$ to (a,b). If we get L along each path, then we say the limit exists and equals to L.
- If there are two paths with different limits, then we say $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

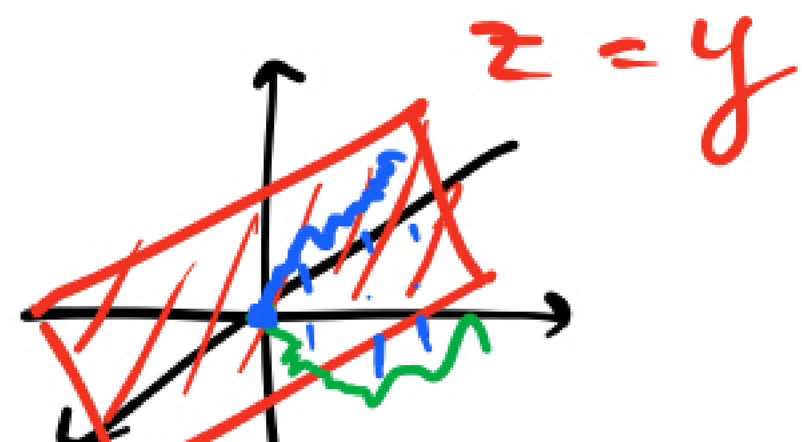
Compare it with Calc 1:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) \text{ exists}$$
$$\lim_{x \to a^+} f(x) \text{ exists}$$
and both equal to L

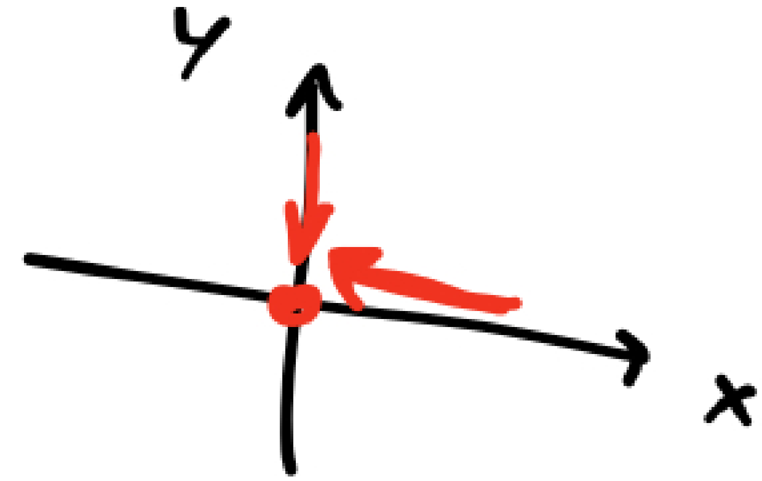
<u>Examples</u> 1 $\displaystyle\lim_{(x,y)\to(0,0)} y = 0$



$y \to 0$ for each path



(2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$



Approach along x-axis

$y=0, \quad x \longrightarrow 0^+$

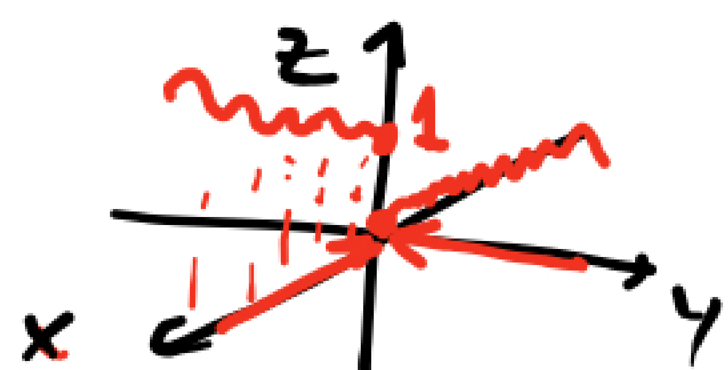
$\lim\limits_{x\to 0^+} \dfrac{x^2}{x^2+0^2} = \lim\limits_{x\to 0^+} 1 = 1.$

Approach along y-axis

$x=0, \quad y \longrightarrow 0^+$

$\lim\limits_{y\to 0^+} \dfrac{0^2}{0^2+y^2} = \lim\limits_{y\to 0^+} 0 = 0.$

Thus lim... DNE

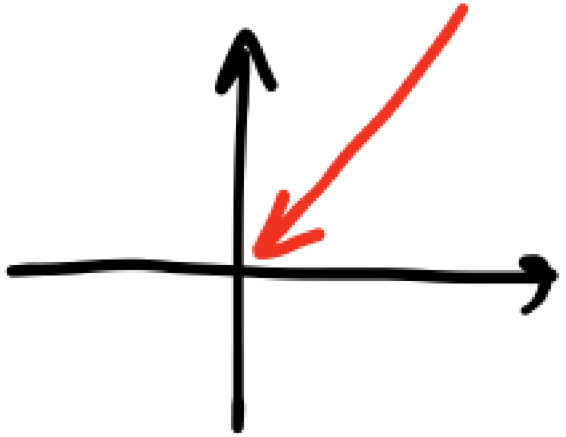


③ $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

along x-axis $(y=0, x\to 0^+)$: $\lim_{x\to 0^+} \frac{x^2-0^2}{x^2+0^2} = 1$

along y-axis $(x=0, y\to 0^+)$: $\lim_{y\to 0^+} \frac{0^2-y^2}{0^2+y^2} = -1.$

DNE

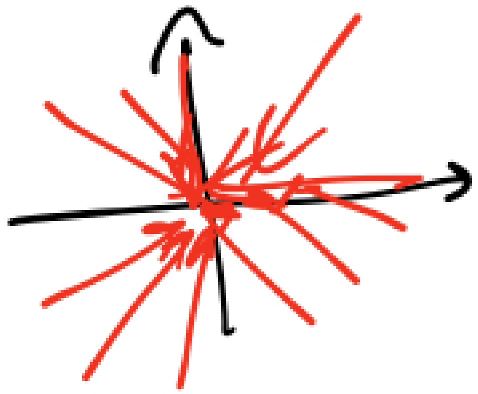


We can also consider the paths $y=mx$, $x\to 0^+$

Along it, the limit is

$$\lim_{x\to 0^+} \frac{x^2-(mx)^2}{x^2+(mx)^2} = \lim_{x\to 0^+} \frac{\cancel{x^2}(1-m^2)}{\cancel{x^2}(1+m^2)} = \frac{1-m^2}{1+m^2}.$$

④ $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2} =$



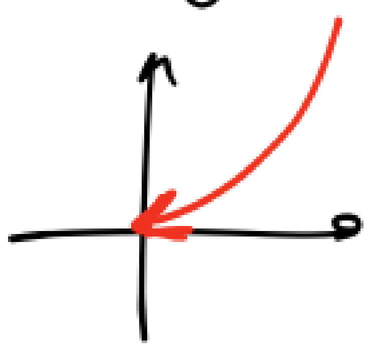
along x-axis: $\lim_{x\to 0^+} \dfrac{0}{x^4+0^2} = 0.$

along y-axis: $\lim_{y\to 0^+} \dfrac{0}{0^4+y^2} = 0.$

along $y=mx$: $\lim_{x\to 0^+} \dfrac{x^2(mx)}{x^4+(mx)^2} =$

$= \lim_{x\to 0^+} \dfrac{mx^{3}}{\cancel{x^2}(x^2+m^2)} = \lim_{x\to 0^+} x \cdot \dfrac{m}{x^2+m^2} = 0$

along $y=x^2$: $x\to 0^+$



$\lim_{x\to 0^+} \dfrac{x^2 \cdot x^2}{x^4+x^4} =$

$= \lim_{x\to 0^+} \dfrac{x^4}{2x^4} = \lim_{x\to 0^+} \dfrac{1}{2} = \dfrac{1}{2}.$ ×

DNE