$$\vec{v} \times \vec{u} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

Properties of cross product:

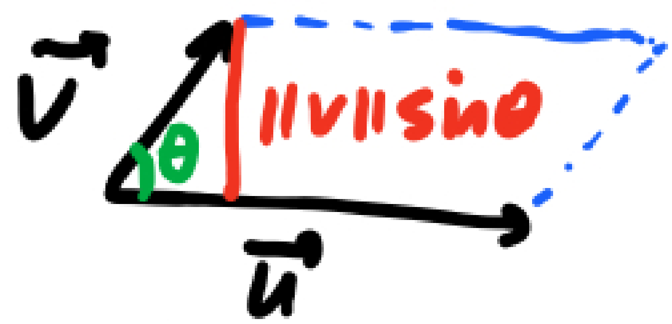
1. $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$

   (in particular, $\vec{v} \times \vec{v} = \vec{0}$)

2. $(c\vec{v}) \times \vec{u} = \vec{v} \times (c\vec{u}) = c(\vec{v} \times \vec{u})$

3. $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$

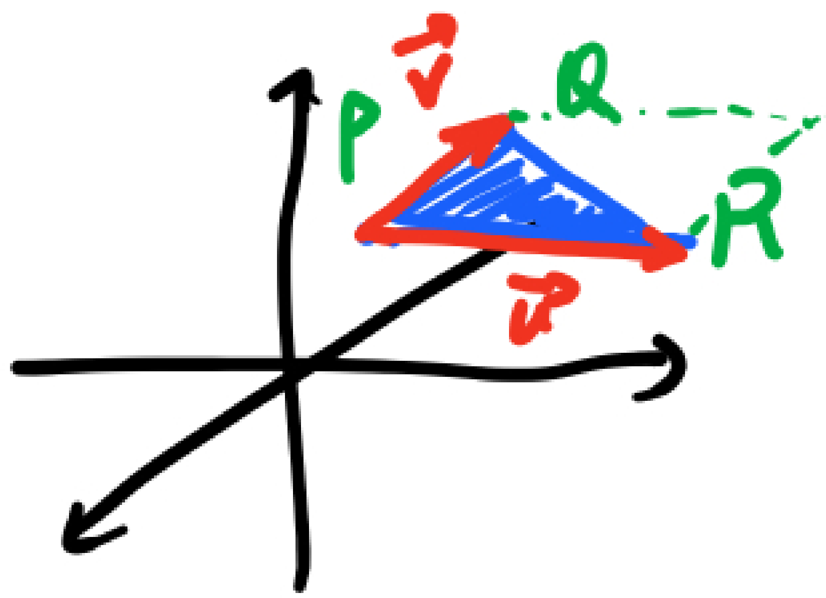
4. $\|\vec{v} \times \vec{u}\| = \|\vec{v}\| \cdot \|\vec{u}\| \sin\theta$



Note $\vec{v} \times (\vec{u} \times \vec{w}) \neq (\vec{v} \times \vec{u}) \times \vec{w}$

Example Find the area of the triangle in $\mathbb{R}^3$ with vertices

$$P = (1,2,1)$$
$$Q = (3,0,0)$$
$$R = (1,1,1).$$



$$\operatorname{area}(\Delta) = \frac{1}{2}\operatorname{area}(\square)$$
$$= \frac{1}{2}\|\vec{v}\times\vec{u}\|$$

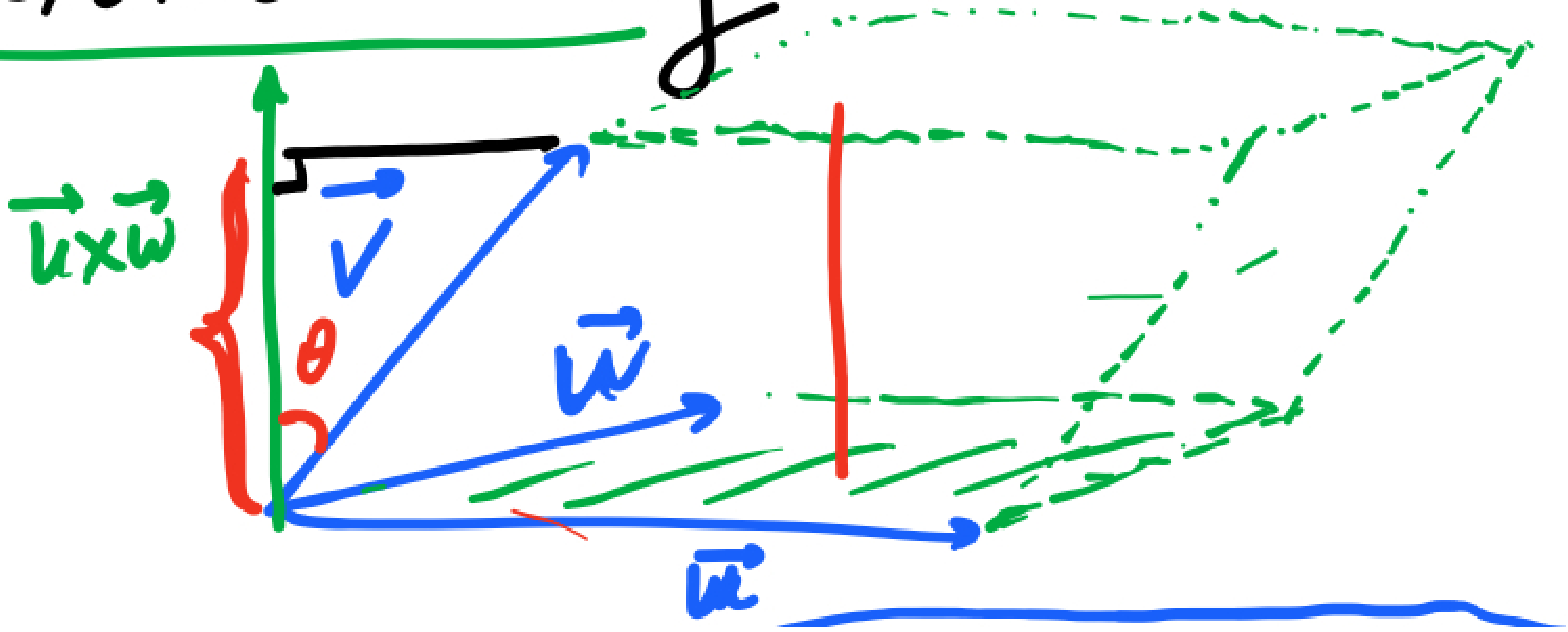
$$\vec{v} = \overrightarrow{PQ} = \vec{Q} - \vec{P} = (2,-2,-1)$$
$$\vec{u} = \overrightarrow{PR} = \vec{R} - \vec{P} = (0,-1,0).$$

$$\vec{v}\times\vec{u} = (-1,0,-2), \quad \operatorname{area}(\Delta) =$$
$$= \frac{1}{2}\underbrace{\|(-1,0,-2)\|}_{\sqrt{(-1)^2+0^2+(-2)^2}} = \boxed{\frac{\sqrt{5}}{2}}$$

Scalar triple product

$$\underbrace{\underbrace{\vec{v}}_{\text{vector}} \cdot \underbrace{(\vec{u} \times \vec{w})}_{\text{vector}}}_{\text{number}} = (\vec{v} \times \vec{u}) \cdot \vec{w}$$

Geometric meaning:



$$\vec{v}\cdot(\vec{u}\times\vec{w}) = \|\vec{v}\|\cdot\underbrace{\|\vec{u}\times\vec{w}\|}_{\text{area of } \square}\cos\theta$$

(dot product formula)

$= (\text{base area})\cdot$ height of parallelepiped $=$

$=$ <u>volume</u>.

<u>upshot</u>: $\left|\vec{v}\cdot(\vec{u}\times\vec{w})\right| =$

$=$ vol of parallelepiped spanned by $\vec{v},\vec{u},\vec{w}$.

$$\boxed{\vec{v}\cdot(\vec{u}\times\vec{w}) = \det\begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}}$$

Observation $\vec{v}\cdot(\vec{u}\times\vec{w})=0$

$\iff$ vol() $= 0$

$\iff$ 

$\vec{v},\vec{u},\vec{w}$ lie on a plane.

Exercise

$$\vec{v} = (1,0,2)$$
$$\vec{u} = (0,2,2)$$
$$\vec{w} = (2,1,5).$$

Show that $\vec{v},\vec{u},\vec{w}$ lie on a plane.