

$$\vec{v} \times \vec{u} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

Properties of cross product:

①  $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$

(in particular,  $\vec{v} \times \vec{v} = \vec{0}$ )

②  $(c\vec{v}) \times \vec{u} = \vec{v} \times (c\vec{u}) = c(\vec{v} \times \vec{u})$

③  $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$

④  $\|\vec{v} \times \vec{u}\| = \|\vec{v}\| \cdot \|\vec{u}\| \sin\theta$

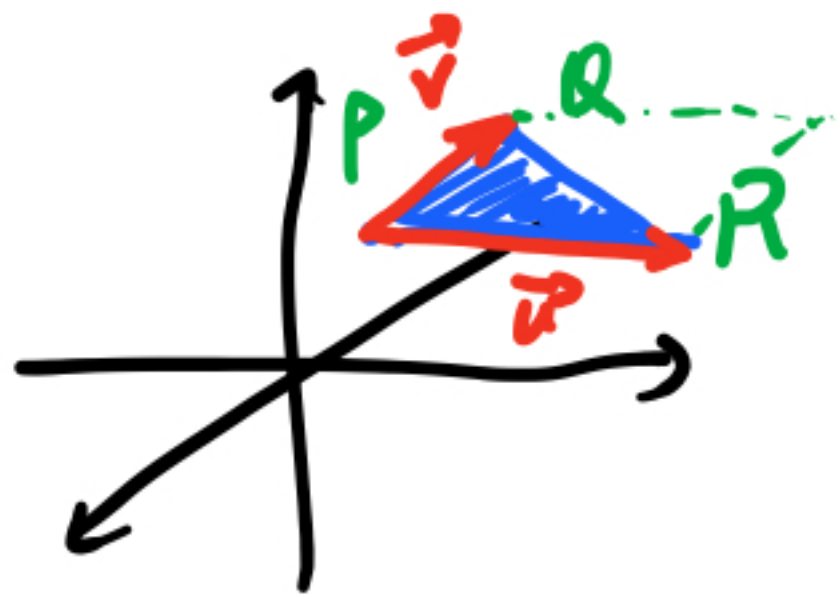


Note  $\vec{v} \times (\vec{u} \times \vec{w}) \neq (\vec{v} \times \vec{u}) \times \vec{w}$

Example Find the area of the triangle  
in  $\mathbb{R}^3$  with vertices  $P = (1, 2, 1)$

$$Q = (3, 0, 0)$$

$$R = (1, 1, 1).$$



$$\text{area}(\Delta) = \frac{1}{2} \text{area}(\square)$$

$$= \frac{1}{2} \|\vec{v} \times \vec{u}\|$$

$$\vec{v} = \overrightarrow{PQ} = \vec{Q} - \vec{P} = (2, -2, -1)$$

$$\vec{u} = \overrightarrow{PR} = \vec{R} - \vec{P} = (0, -1, 0).$$

$$\vec{v} \times \vec{u} = (-1, 0, -2), \text{ area}(\Delta) =$$

$$= \frac{1}{2} \|(-1, 0, -2)\| =$$

$$\frac{1}{2} \sqrt{(-1)^2 + 0^2 + (-2)^2}$$

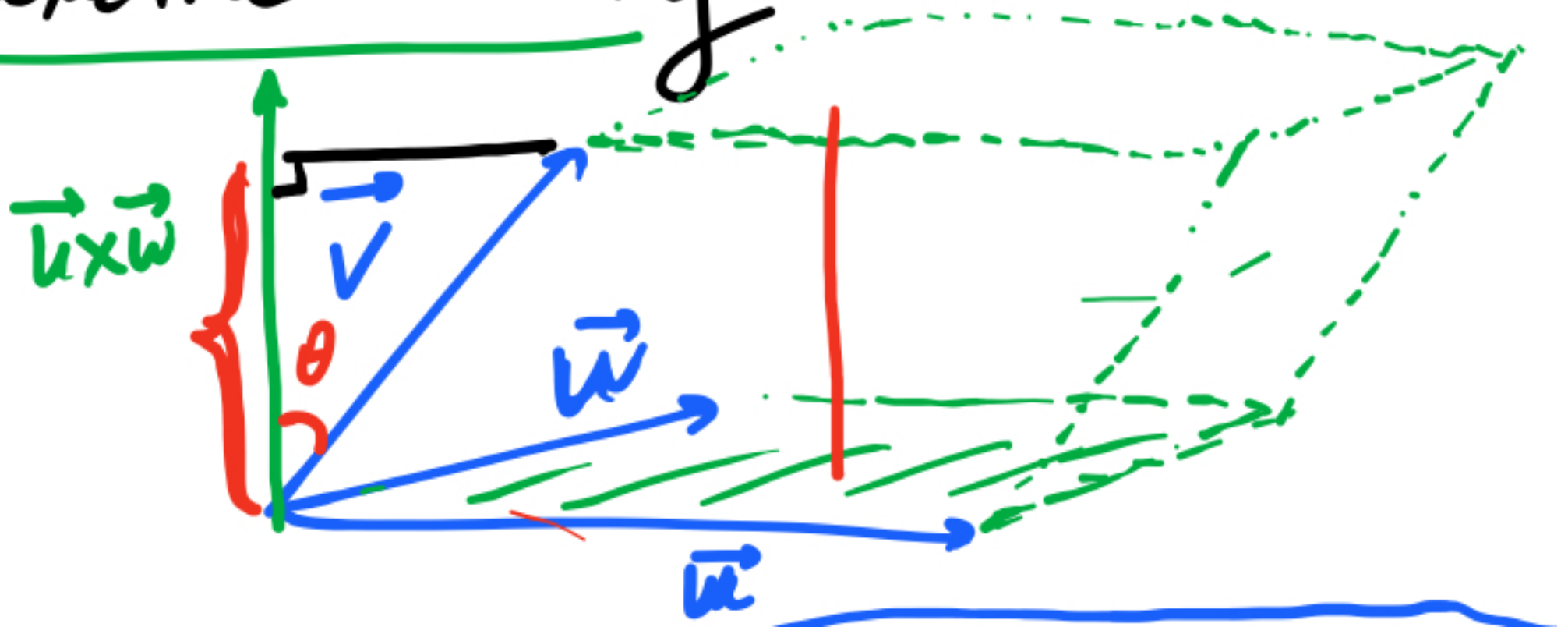
$$\boxed{\frac{\sqrt{5}}{2}}$$

# Scalar triple product

$$\underbrace{\vec{v} \cdot (\vec{u} \times \vec{w})}_{\text{number}} = (\vec{v} \times \vec{u}) \cdot \vec{w}$$

The diagram shows the equation  $\vec{v} \cdot (\vec{u} \times \vec{w}) = (\vec{v} \times \vec{u}) \cdot \vec{w}$ . Red annotations include: a bracket under  $\vec{u} \times \vec{w}$  labeled "vector", a bracket under  $\vec{v}$  labeled "vector", and a large bracket under the entire left-hand side labeled "number".

Geometric meaning:



$$\vec{v} \cdot (\vec{u} \times \vec{w}) = \|\vec{v}\| \cdot \|\vec{u} \times \vec{w}\| \cos \theta$$

(dot product formula)

area of 

= (base area) · height of parallelepiped =  
= volume .

upshot:  $|\vec{v} \cdot (\vec{u} \times \vec{w})| =$

= vol of parallelepiped spanned  
by  $\vec{v}, \vec{u}, \vec{w}$ .

$$\vec{v} \cdot (\vec{u} \times \vec{w}) = \det \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Observation  $\vec{v} \cdot (\vec{u} \times \vec{w}) = 0$

$\Leftrightarrow \text{vol}(\text{[cube]}) = 0$

$\Leftrightarrow$    $\vec{v}, \vec{u}, \vec{w}$   
lie on a plane.

Exercise  $\vec{v} = (1, 0, 2)$

$$\vec{u} = (0, 2, 2)$$

Show  
that

$$\vec{w} = (2, 1, 5).$$

$\vec{v}, \vec{u}, \vec{w}$  lie on a plane.