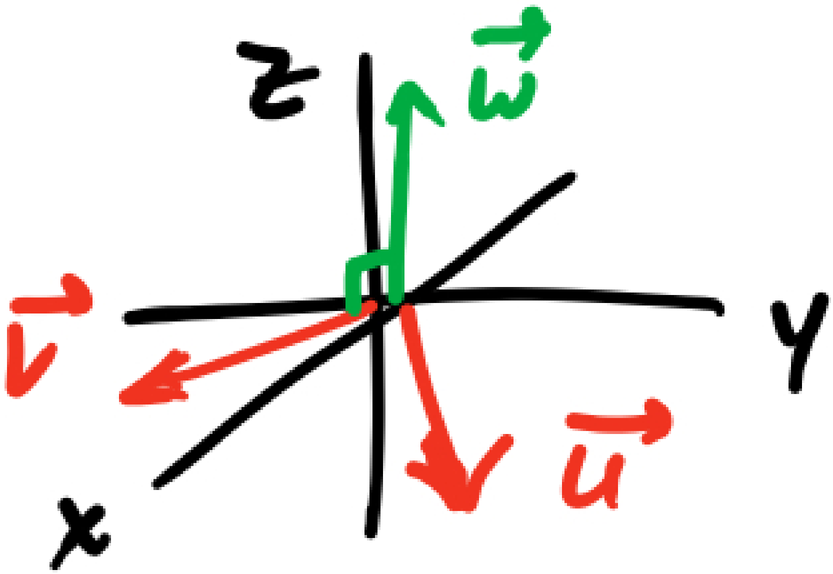
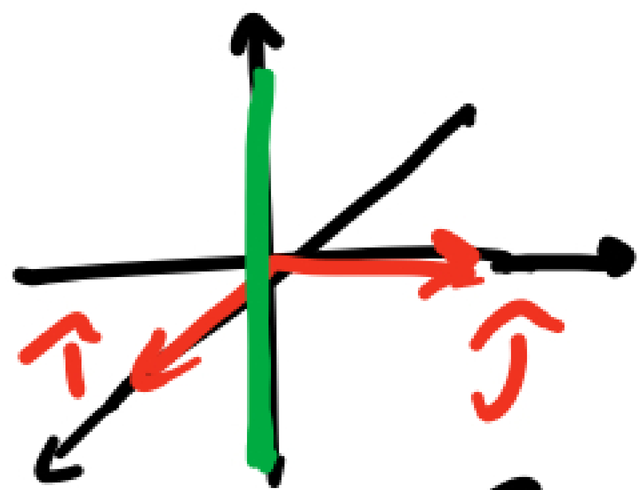
Cross product.
(only in $\mathbb{R}^3$!)

Problem

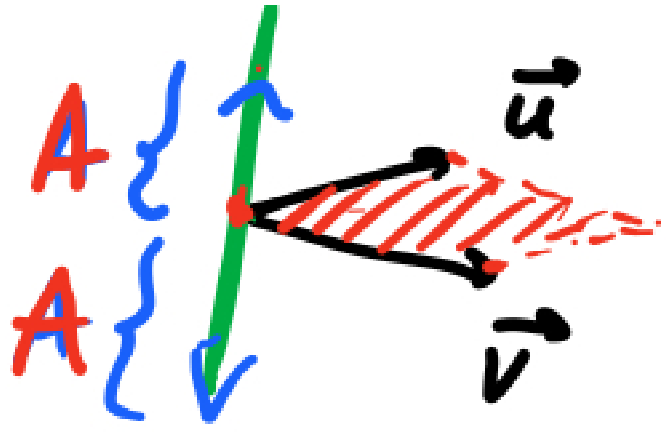


Find a vector $\vec{w}$ perp to both $\vec{v}$ and $\vec{u}$.

Example

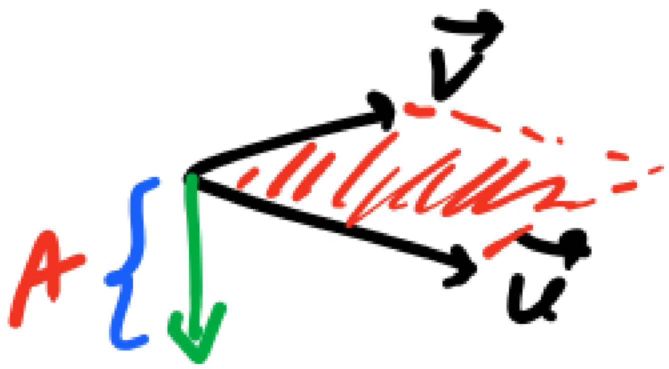


which vector to choose?



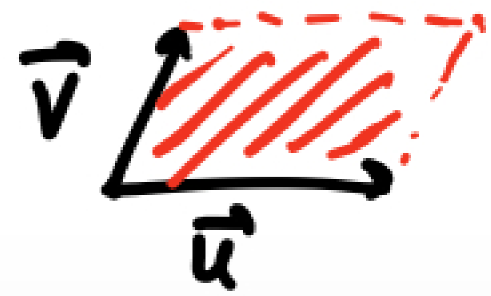
A - area of parallelogram

+ right-hand rule



(so $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$)

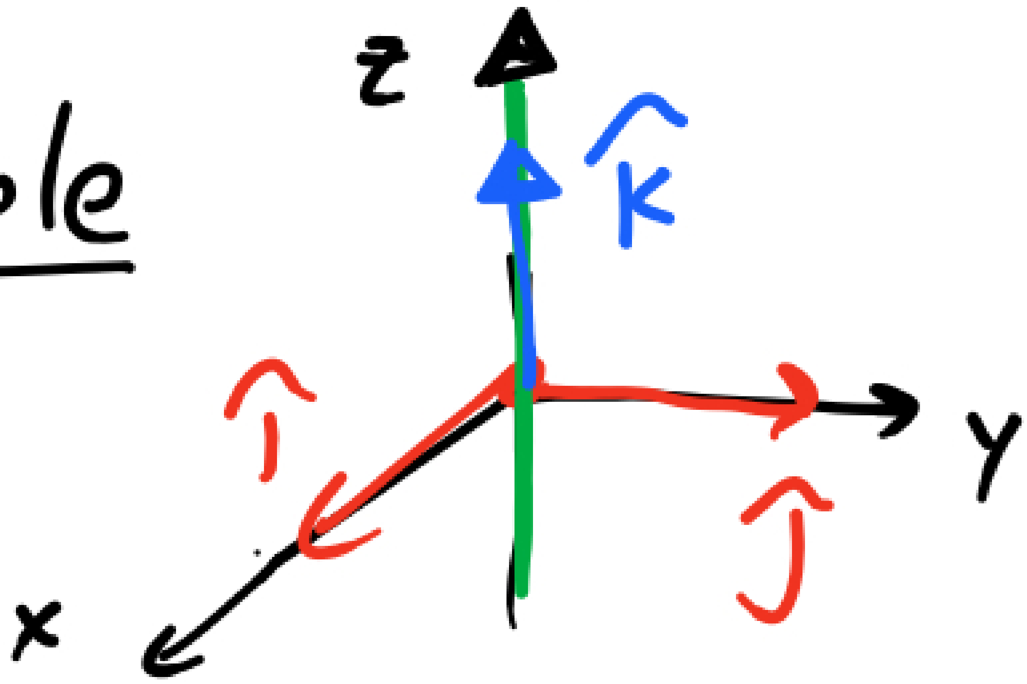
Definition The cross product of $\vec{v}$ and $\vec{u}$ ($\vec{v}, \vec{u} \in \mathbb{R}^3$) is the vector $\vec{v} \times \vec{u}$ that is perp. to both $\vec{v}$ and $\vec{u}$, whose length is equal to the area of parallelogram spanned by $\vec{v}$ and $\vec{u}$,



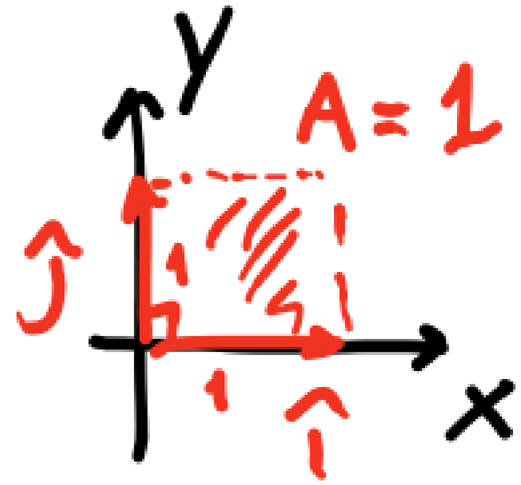
and whose direction is determined by the right-hand rule.

How to compute cross product?

Example



$\hat{i} \times \hat{j}$



$$\hat{i} \times \hat{j} = \hat{k}$$

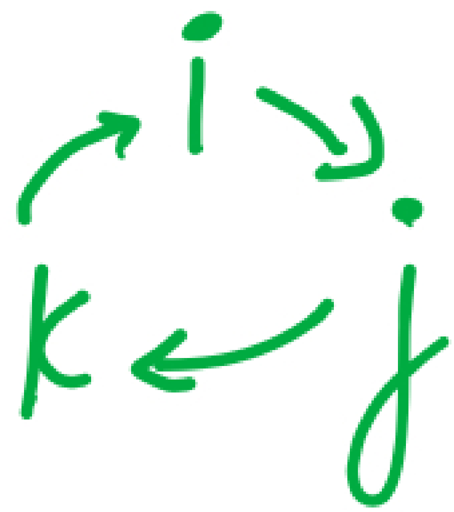
$$(\hat{j} \times \hat{i} = -\hat{k})$$

Similarly,

$$\hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j}$$

Mnemonic rule:



2×2 and 3×3 determinants

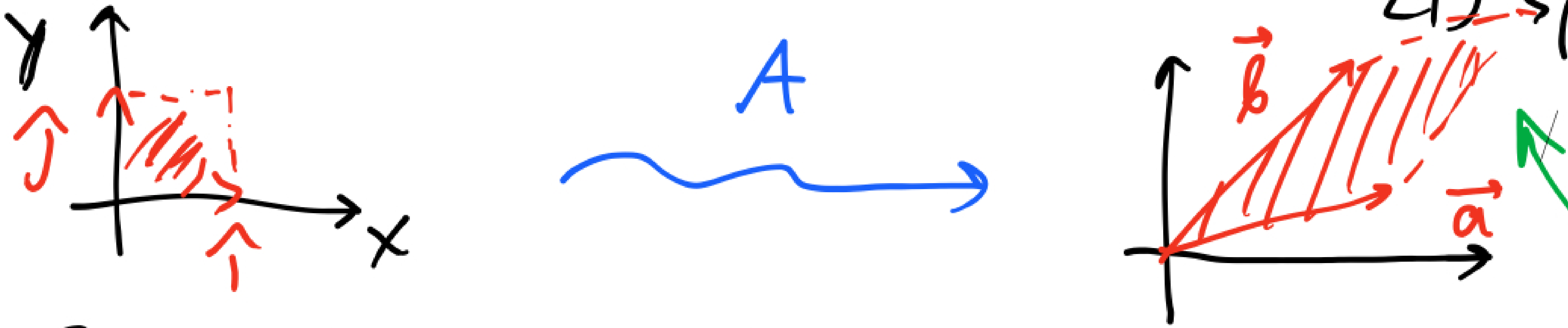
An $n \times m$ matrix is an array of real numbers with n rows and m columns

$$A = n\left\{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & & & a_{nm} \end{bmatrix}\right. \quad \underbrace{\hphantom{a_{11}\ a_{12}\ a_{13}\ \cdots\ a_{1m}}}_{m}$$

2×2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $a,b,c,d \in \mathbb{R}$

3×3: $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$, $a_i \in \mathbb{R}$

→ Can be thought of "transformations" of 2D space



The <u>determinant</u> of A is the (signed) area of parallelogram above

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det A = \underline{ad - bc}$.

If $A = \begin{bmatrix} \boxed{a_1 \quad a_2 \quad a_3} \\ a_4 \quad a_5 \quad a_6 \\ a_7 \quad a_8 \quad a_9 \end{bmatrix}$, then $\det A =$

$$= a_1 \det \begin{bmatrix} a_5 & a_6 \\ a_8 & a_9 \end{bmatrix} - a_2 \det \begin{bmatrix} a_4 & a_6 \\ a_7 & a_9 \end{bmatrix} + a_3 \det \begin{bmatrix} a_4 & a_5 \\ a_7 & a_8 \end{bmatrix}$$

<u>Example</u>

$$\det \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 5 \end{bmatrix} = 1 \det \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} - (-2) \det \begin{bmatrix} 0 & 1 \\ 2 & 5 \end{bmatrix} + 0 \det \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} =$$

$$= 1 \cdot 4 + 2 \cdot (-2) + 0 = \underline{0}.$$

Back to cross products

- If $\vec{v} = (v_1, v_2, v_3)$, $\vec{u} = (u_1, u_2, u_3)$, then

$$\vec{v} \times \vec{u} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{bmatrix} =$$

abuse of notation

$$= \vec{i} \det \begin{bmatrix} v_2 & v_3 \\ u_2 & u_3 \end{bmatrix} - \vec{j} \det \begin{bmatrix} v_1 & v_3 \\ u_1 & u_3 \end{bmatrix} + \vec{k} \det \begin{bmatrix} v_1 & v_2 \\ u_1 & u_2 \end{bmatrix}.$$

Examples

- $\hat{\imath} \times \hat{\jmath} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} =$

$$= \hat{\imath} \det \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \hat{\jmath} \det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \hat{k} \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \hat{k}.$$

- $(0, -1, 0) \times (2, -2, -1) = ?$

$$\hat{\imath} + 0\hat{\jmath} + 2\hat{k} = (1, 0, 2).$$