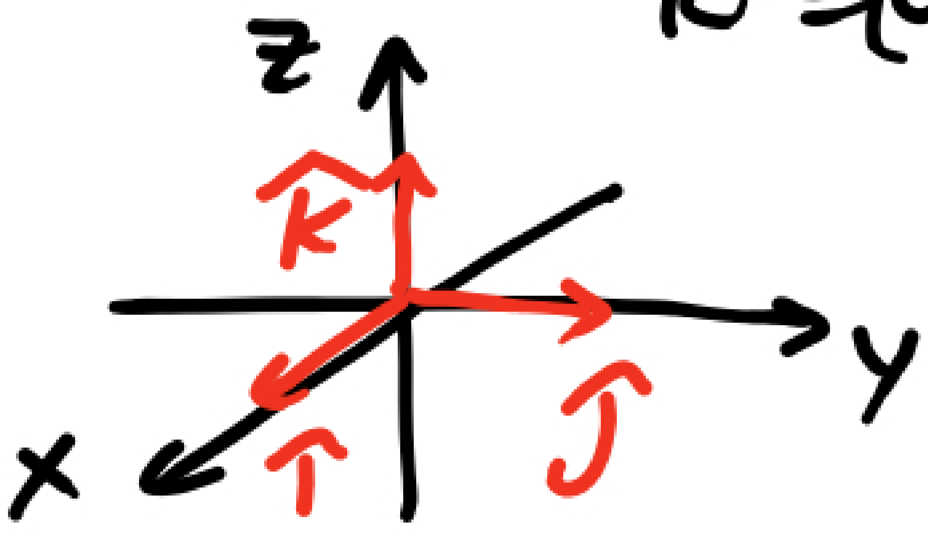
Dot product

Last time: $\vec{v} \cdot \vec{u} = v_1 u_1 + v_2 u_2 + v_3 u_3$

- $v \cdot v = \|v\|^2$
- $v \cdot u = \|v\| \cdot \|u\| \cdot \cos\theta$
- $v \cdot u = 0 \iff v \perp u$

<u>Example</u>

$$\left.\begin{aligned} \hat{\imath} &= (1,0,0) \\ \hat{\jmath} &= (0,1,0) \\ \hat{k} &= (0,0,1) \end{aligned}\right\} \text{"Standard basis for } \mathbb{R}^3\text{"}$$



Any vector $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ can be represented in terms of $\hat{\imath}, \hat{\jmath}, \hat{k}$:

$\rightarrow$

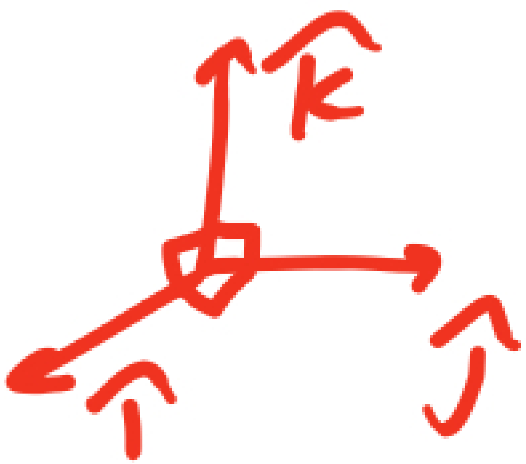
$$\vec{v} = (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) =$$

$$= v_1 (1,0,0) + v_2 (0,1,0) + v_3 (0,0,1) =$$

$$= \underline{v_1}\, \hat{\imath} + v_2 \hat{\jmath} + v_3 \hat{k}.$$

Note that $\|\hat{\imath}\| = \|\hat{\jmath}\| = \|\hat{k}\| = 1$.

(<u>unit</u> vectors).



$$\hat{\imath} \cdot \hat{\jmath} = (1,0,0) \cdot (0,1,0) = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0.$$

$$\Rightarrow \hat{\imath} \perp \hat{\jmath}.$$

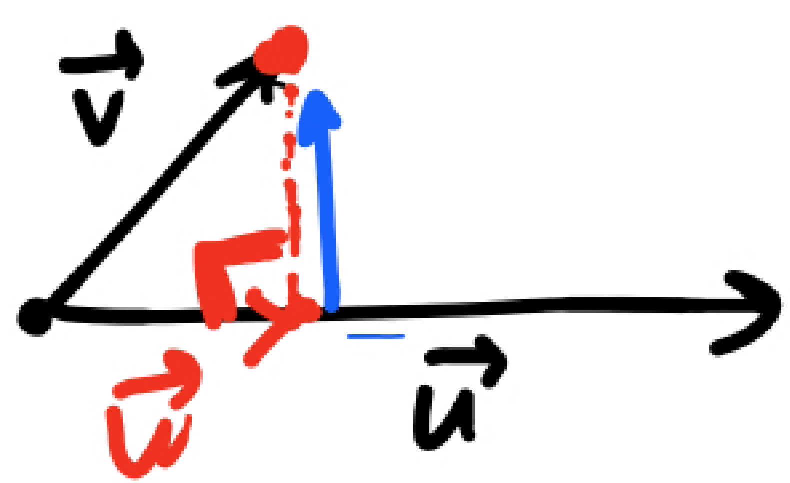
Similarly, $\hat{\imath} \perp \hat{k}$, $\hat{\jmath} \perp \hat{k}$.

Vector division? In general, NO

If $\vec{v}\cdot\vec{u} = \vec{v}\cdot\vec{w}$ and $\vec{v}\neq 0$

$\not\Rightarrow$ $\vec{u}=\vec{w}$ in general.

Projections



Find $\vec{w}$ — projection of $\vec{v}$ onto $\vec{u}$

$\vec{w}$ points in direction of $\vec{u}$:

$$\vec{w} = c\,\vec{u}$$

we have $$(\vec{v}-\vec{w})\cdot\vec{u} = 0.$$

$$\vec{v}\cdot\vec{u} - \vec{w}\cdot\vec{u} = 0$$

$$\vec{v}\cdot\vec{u} = \underline{\vec{w}}\cdot\vec{u}$$

$$\Rightarrow \underbrace{\vec{v}\cdot\vec{u}}_{\text{number}} = \overline{c}\underbrace{(\vec{u}\cdot\vec{u})}_{\text{number}} \Rightarrow c = \frac{v\cdot u}{u\cdot u}$$

Hence $\vec{w} = \dfrac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\, u$

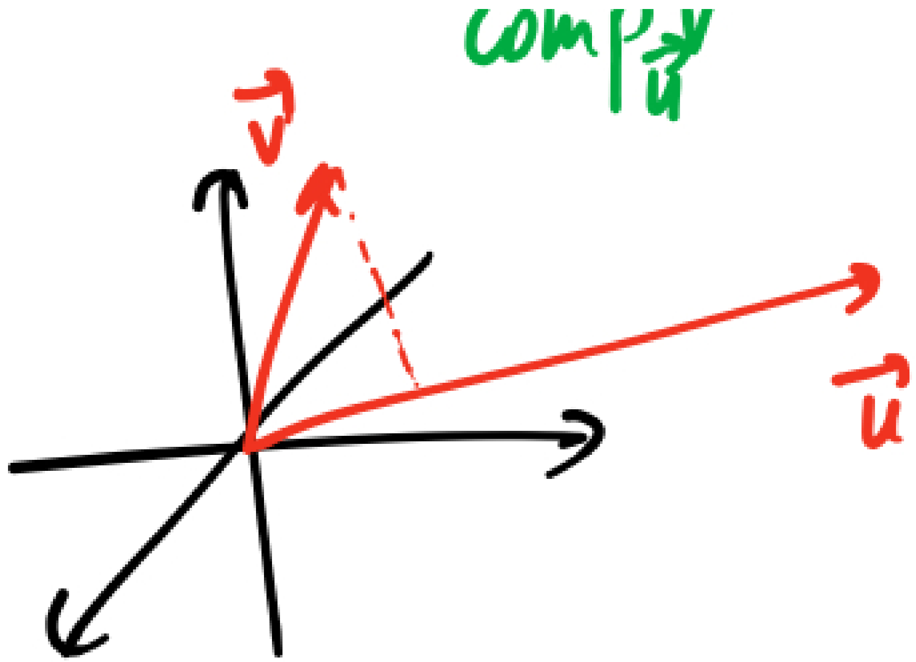
$$\boxed{\operatorname{Proj}_{\vec{u}}\vec{v} = \frac{\vec{v}\cdot\vec{u}}{\|u\|^2}\vec{u}} = \frac{\vec{v}\cdot\vec{u}}{\|u\|}\,\frac{\vec{u}}{\|u\|}$$

$$= \underbrace{\frac{v\cdot u}{\|u\|}}_{\operatorname{comp}_{\vec{u}}\vec{v}}\,\hat{u}$$

Example

$\vec{v} = (1,2,3)$.

What is $\text{proj}_{\hat{\imath}}\vec{v}$?



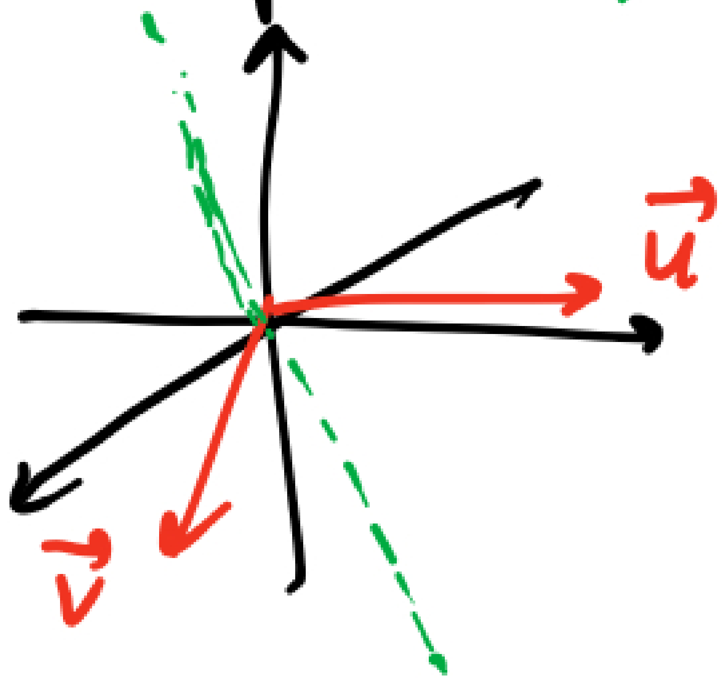
$$\text{proj}_{\hat{\imath}}\vec{v} = \frac{\vec{v}\cdot\hat{\imath}}{\|\hat{\imath}\|^2}\hat{\imath} = \frac{\overset{1\cdot 1+2\cdot 0+3\cdot 0}{(1,2,3)\cdot(1,0,0)}}{1^2}\hat{\imath} = \hat{\imath}.$$

Similarly $\text{proj}_{\hat{\jmath}}\vec{v} = 2\hat{\jmath}$.

$\text{proj}_{\hat{k}}\vec{v} = 3\hat{k}$.

12.4. Cross product (ONLY IN $\mathbb{R}^3$)

Problem



Find $\vec{w}$ that is perp to $\vec{v}$ and $\vec{u}$ (in $\mathbb{R}^3$)

Idea: want $\vec{w} \perp \vec{v}$ and $\vec{w} \perp \vec{u}$

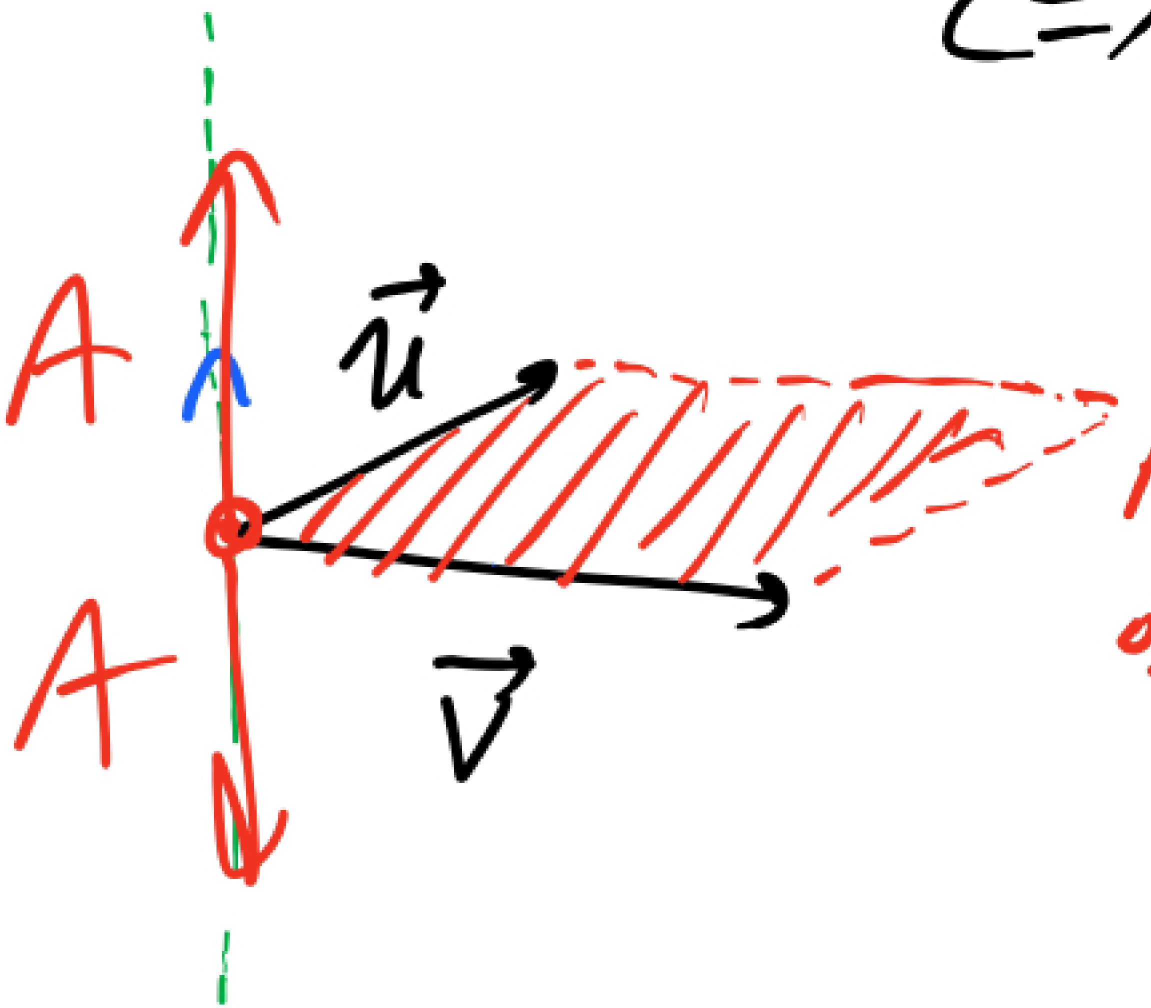
$$\Leftrightarrow \begin{cases} \vec{w} \cdot \vec{v} = 0 \\ \vec{w} \cdot \vec{u} = 0 \end{cases}$$ solve for $\vec{w}$

Example Find $\vec{w} \perp \hat{\imath}, \hat{\jmath}$.

e.g. $\vec{w} = \hat{k}$.

$$\begin{cases} \vec{w} \cdot \hat{\imath} = 0 \\ \vec{w} \cdot \hat{\jmath} = 0 \end{cases} \iff \begin{cases} (w_1, w_2, w_3) \cdot (1,0,0) = 0 \\ (w_1, w_2, w_3) \cdot (0,1,0) = 0 \end{cases}$$

$$\iff \begin{cases} w_1 = 0 \\ w_2 = 0 \end{cases} \quad \vec{w} = (0,0,w_3).$$



A = area of parallelogram