8. Finite Products

1 Motivation

This is the last part in our series exploring how to get new topological spaces from old ones. For now, at least. We mentioned the definition of the product topology for a finite product way back in Example 2.3.6 in the lecture notes concerning bases of topologies, but we did not do anything with it at the time. In that same section we also discussed the following basis on $\mathbb{R}^2$ that we were eventually able to show generates the usual topology on $\mathbb{R}^2$:

$$\mathcal{B} = \left\{ (a,b) \times (c,d) \subseteq \mathbb{R}^2 \,:\, a < b,\, c < d \right\}.$$

This is, essentially, how (finite) product topologies work in general, as we will see shortly.

We will also explore a new way of analyzing topological properties themselves. We have already seen that all the topological properties we care about are preserved by homeomorphisms, and that some are preserved under weaker maps like continuous surjections (recall that the image of a dense set under a continuous function is dense in the range of the function). We also saw that some topological properties are hereditary (like Hausdorffness and second countability) while some are not (like separability). In this section we will explore another way of analyzing properties, by asking whether they are preserved by finite products.

Also, while reading this section, note that we are specifically talking about *finite* products of topological spaces, and any time we refer to a product of spaces the reader should assume we mean a finite product. Infinite products are substantially more complicated, and we will deal with them later in the course. Finite products are actually quite straightforward.

2 Finite product topologies

Definition 2.1. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. The product topology on $X \times Y$ is the topology generated by the basis*

$$\{ U \times V \,:\, U \in \mathcal{T},\, V \in \mathcal{U} \}.$$

More generally if $(X_1, \mathcal{T}_1), \ldots, (X_n, \mathcal{T}_n)$ are topological spaces, the product topology on

$$\prod_{i=1}^{n} X_i = X_1 \times \cdots \times X_n$$

is the topology generated by the basis

$$\{ U_1 \times U_2 \times \cdots \times U_n \,:\, U_i \in \mathcal{T}_i \text{ for all } i = 1, \ldots, n \}.$$

Simple as that. This definition is what you should want it to be, more or less. It is natural to expect that products of sets that are open in each coordinate should be open in the product. That alone does not give you a topology, it turns out, but it does give you a basis. So you generate a topology from it, and the result is the product topology.

As you would expect, bases play nicely with this definition as well, as shown by the following easy proposition.

Proposition 2.2. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be bases on X and Y that generate $\mathcal{T}$ and $\mathcal{U}$, respectively. Then*

$$\mathcal{B} = \{ U \times V \, : \, U \in \mathcal{B}_X, \, V \in \mathcal{B}_Y \} .$$

is a basis for the product topology on $X \times Y$.

Proof. $\mathcal{B}_X \subseteq \mathcal{T}$ and $\mathcal{B}_Y \subseteq \mathcal{U}$, and so every element of $\mathcal{B}$ is open in the product topology.

Now fix an open set U in the product topology, and some point $(x, y) \in U$. We need to find an element $B \in \mathcal{B}$ such that $x \in B \subseteq U$. By definition of the product topology, there must be some $U_X \in \mathcal{T}$ and $U_Y \in \mathcal{U}$ such that $(x, y) \in U_X \times U_Y \subseteq U$. Using the fact that $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases, find sets $B_X \in \mathcal{B}_X$ and $B_Y \in \mathcal{B}_Y$ such that $x \in B_X \subseteq U_X$ and $y \in B_Y \subseteq U_Y$. But then we have:

$$(x, y) \in B_X \times B_Y \subseteq U_X \times U_Y \subseteq U,$$

so $B = B_X \times B_Y$ is the set we were looking for. □

Of course, this fact generalizes to larger finite products and the proof is similarly straightforward.

Remark 2.3. As a matter of notation, we will usually write X^2 instead of $X \times X$, X^3 instead of $X \times X \times X$, and so on. This agrees with the usual notation for $\mathbb{R}^n$.

Before going on, here are some simple examples.

Example 2.4.

1. A product of discrete spaces is discrete, and a product of indiscrete spaces is indiscrete.

2. $(\mathbb{R}_{\text{usual}})^2 = \mathbb{R}^2_{\text{usual}}$.

3. $(\mathbb{R}_{\text{Sorgenfrey}})^2$ is an interesting space. This is the space generated by the basis of rectangles with their left and bottom edges closed. You will explore this space more through some Big List problems.

3 Projections

We have mentioned projections functions from $\mathbb{R}^2$ to $\mathbb{R}$ already, but they have an important relationship to products of topological spaces in general.

Definition 3.1. *Let $(X_1, \mathcal{T}_1), \dots, (X_n, \mathcal{T}_n)$ be topological spaces. Define the projection maps*

$$\pi_k : \prod_{i=1}^{n} X_i \to X_k$$

for $k = 1, \dots, n$ by $\pi_k(x_1, \dots, x_n) = x_k$.

For the purposes of writing proofs in this section, we will restrict our discussion to products of two spaces to avoid unnecessary complication and indexing. Everything we say here will extend to all finite products in the obvious ways.

So for two spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$, the projection functions are $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ given by

$$\pi_1(x, y) := x \quad \text{and} \quad \pi_2(x, y) := y.$$

The following fact will be of use to us. If these facts are not obvious to you from the definition of the projection functions, draw yourself a picture and they will become obvious.

Fact 3.2. *Let $A \subseteq X$ and $B \subseteq Y$. Then $\pi_1^{-1}(A) = A \times Y$, and $\pi_2^{-1}(B) = X \times B$. Moreover, $A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B)$,*

Projection functions arise naturally when you discuss Cartesian products of sets. (Notice that the preceding fact does not mention topologies at all.) The following proposition is a way of characterizing the product topology on $X \times Y$ in terms of the continuity of these projection functions. At the moment it will seem unwieldy compared to the definition we gave above, but this characterization will be very useful for us when we discuss infinite products later in the course.

Proposition 3.3. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. Then the product topology on $X \times Y$ is the coarsest topology on $X \times Y$ such that the projections π_1 and π_2 are continuous.*

Proof. By the fact above it is easy to see that the projection functions are continuous in the product topology, so it only remains to show that the product topology is the coarsest topology with this property.

So let $\mathcal{V}$ be a topology on $X \times Y$ such that π_1 and π_2 are continuous. We will show that $\mathcal{V}$ refines the product topology by showing that $U_X \times U_Y \in \mathcal{V}$ for every $U_X \in \mathcal{T}$ and $\mathcal{U}_Y \in \mathcal{U}$.

So fix such a set $U_X \times U_Y$. Then $\pi_1^{-1}(U_X) = U_X \times Y$ and $\pi_2^{-1}(U_Y) = X \times U_Y$ are in $\mathcal{V}$ by assumption. Since $\mathcal{V}$ is a topology, the intersection of these two sets must also be in $\mathcal{V}$. That is:

$$(U_X \times Y) \cap (X \times U_Y) = U_X \times U_Y \in \mathcal{V},$$

as required. □

A slightly different way of seeing the previous result is that if π_1 and π_2 are to be continuous, then $U_X \times Y$ and $X \times U_Y$ must be open in the product for all $U_X \in \mathcal{T}$ and $U_Y \in \mathcal{U}$, since these are the preimages of open sets under these two maps. These sets obviously cover the product (since the sets in $\mathcal{T}$ and $\mathcal{U}$ cover X and Y, respectively), and therefore form a subbasis on $X \times Y$. The basis they generate is precisely $\{\, U \times V \,:\, U \in \mathcal{T},\ V \in \mathcal{U} \,\}$ (the usual basis for the product topology), by the fact above. Therefore any topology in which the projections are continuous must contain this basis, and therefore must refine the product topology.

Just to state this for posterity:

Proposition 3.4. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. Then the set*

$$\begin{aligned} \mathcal{S} &:= \{\, \pi_1^{-1}(U) \,:\, U \in \mathcal{T} \,\} \cup \{\, \pi_2^{-1}(V) \,:\, V \in \mathcal{U} \,\} \\ &= \{\, U \times Y \,:\, U \in \mathcal{T} \,\} \cup \{\, X \times V \,:\, V \in \mathcal{U} \,\} \end{aligned}$$

is a subbasis that generates the product topology on $X \times Y$.

The main use for the projection functions we have right now is in characterizing continuous functions to product topologies from other spaces. It turns out that they work exactly the way you wish they would.

This is a result you should be familiar with in the context of multivariable calculus at least.

Proposition 3.5. *Let $(X, \mathcal{T})$, $(Y_1, \mathcal{U}_1)$, and $(Y_2, \mathcal{U}_2)$ be topological spaces, and let $f : X \to Y_1 \times Y_2$ be a function. Then f is continuous if and only if $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.*

Proof. ($\Rightarrow$). This follows from the fact that a composition of continuous functions is continuous.

($\Leftarrow$). Suppose $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous. We will show that the preimage of a subbasic open subset of $Y_1 \times Y_2$ like the ones we described above is open. Let $U \times Y_2$ be such a subbasic open set. Then:

$$f^{-1}(U \times Y_2) = f^{-1}(\pi_1^{-1}(U)) = (\pi_1 \circ f)^{-1}(U),$$

which is open since $\pi_1 \circ f$ is continuous. The case of subbasic open sets of the form $Y_1 \times V$ is analogous. □

4 Finitely productive properties

As promised, here is another way of analyzing topological properties.

Definition 4.1. *A property ϕ of topological spaces is said to be finitely productive if every finite product of topological spaces with ϕ has ϕ.*

That is, if $(X_1, \mathcal{T}_1), \ldots, (X_n, \mathcal{T}_n)$ all have ϕ, then $\prod_{i=1}^{n} X_i$ with the product topology has ϕ.

At the moment, most (but not all) of the topological properties we have studied are finitely productive. For example, all of the following properties are finitely productive. As with the proofs that the various properties we listed in the previous set of notes were hereditary, the proofs here are all purely unwinding definitions. We will do one proof as an example.

1. T_0 and T_1.
2. Hausdorff.
3. Finite.
4. Countable (more generally, any infinite cardinality).
5. Separable.
6. First countable.
7. Second countable.

Proof that the Hausdorff property is finitely productive. As usual, it suffices to show this for a product of two spaces. So let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be Hausdorff topological spaces, and let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be distinct points in $X \times Y$. Then it must be that $x_1 \neq x_2$ or $y_1 \neq y_2$ (or both). Without loss of generality, assume $x_1 \neq x_2$.

Since $(X, \mathcal{T})$ is Hausdorff, let U_1, U_2 be disjoint open subsets of X such that $x_1 \in U_1$ and $x_2 \in U_2$. But then $U_1 \times Y$ and $U_2 \times Y$ are disjoint open subsets of $X \times Y$ containing z_1 and z_2, respectively. Therefore, $X \times Y$ is Hausdorff. □

In the next set of notes (on the stronger separation axioms) we will see an interesting property which is not finitely productive. For the moment though, there is one topological property we have studied that is notably missing from the list above: the countable chain condition.

It is a somewhat surprising fact that the question of whether the product of two ccc spaces is ccc is independent of the usual axioms of mathematics. A theorem of Rich Laver's from the 1970s showed that if you assume the Continuum Hypothesis, then you can construct two ccc topological spaces whose product is not ccc. On the other hand, if you assume an independent axiom of set theory called Martin's Axiom, an arbitrary product of ccc spaces is ccc. The proofs of these facts are well outside the scope of this course, though interested students are encouraged to do some research into them.

5 A little preview of arbitrary products

As we said in the first section, arbitrary products of topological spaces are a little weirder than finite products. This is mainly because the two characterizations we gave (Definition 2.1 and Proposition 3.3) disagree in this new context. The definition in terms of projection functions turns out to be the "correct" one.

To see that they are different, consider the particular case of a countably infinite product of copies of $\mathbb{R}_{\text{usual}}$. We will refer to the underlying set as $\mathbb{R}^{\mathbb{N}}$.

By analogy with with Definition 2.1, we would define a topology on $\mathbb{R}^{\mathbb{N}}$ by defining the basis

$$\mathcal{B} = \left\{ \prod_{n\in\mathbb{N}} U_n \, : \, U_n \text{ open in } \mathbb{R}_{\text{usual}} \right\}.$$

So for example in this topology, the set $(0,1) \times (0,1) \times (0,1) \times \cdots$ would be a basic open set.

By analogy with Proposition 3.3 we would define the same subbasis $\mathcal{S}$ (which you can check is still a subbasis in this context), and get subbasic open sets of the form:

$$\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times U \times \mathbb{R} \times \cdots,$$

where U is an open set in $\mathbb{R}_{\text{usual}}$. Finite intersections of elements of this subbasis would form a basis, as usual. However we can see that in a finite intersection of sets of this form, only finitely many coordinates will have sets other than $\mathbb{R}$. Stated another way, all but finitely many coordinates will be $\mathbb{R}$. In particular, the set $(0,1) \times (0,1) \times \cdots$ mentioned above would not be open in this topology.

It turns out that the second definition is the one we want for arbitrary products due to some convenient properties it has, such as a function being continuous if and only if all of its coordinate functions are continuous (the analogue of Proposition 3.5), sequences converging if and only if all the coordinate sequences converge, etc. This second definition is called the product topology, while the first one (generated by the basis $\mathcal{B}$) is called the box topology.

As a warm up exercise for next term, try to prove yourself that the following sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{N}}$ converges in the product topology, but not in the box topology:

$$\begin{aligned} x_1 &= (1,1,1,1,1,\dots) \\ x_2 &= (0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\dots) \\ x_3 &= (0,0,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\dots) \\ x_4 &= (0,0,0,\tfrac{1}{4},\tfrac{1}{4},\dots) \end{aligned}$$