Conformality

Angle-Preserving Linear Maps and Conformal Functions

Lecture Notes

October 15, 2026

1 Angle-Preserving Linear Maps

Definition (angle-preserving)

Let $T : \mathbb{C} \to \mathbb{C}$ be an $\mathbb{R}$-linear invertible map. T is **angle-preserving** if

$$\forall w, z \in \mathbb{C} : \quad |w||z|\langle Tw, Tz\rangle = |Tw||Tz|\langle w, z\rangle,$$

where $\langle w, z\rangle := \operatorname{Re} w \operatorname{Re} z + \operatorname{Im} w \operatorname{Im} z = \operatorname{Re}(z\bar{w})$ is the usual scalar product, so that $\dfrac{\langle w, z\rangle}{|w||z|} = \cos\theta$ for $\theta = \angle_w^z$, the angle between z and w.

Examples. $Tz = az$ (**complex-linear**) and $Tz = a\bar{z}$ (**complex-antilinear**), for fixed $a \in \mathbb{C} \setminus \{0\}$.

Lemma (angle-preserving linear maps)

The following are equivalent for an $\mathbb{R}$-linear map $T : \mathbb{C} \to \mathbb{C}$:

(1) T is angle-preserving.

(2) Either $\exists\, a \in \mathbb{C} \setminus \{0\}$ with $Tz = az\ \forall z$ (complex-linear), or $\exists\, a \in \mathbb{C} \setminus \{0\}$ with $Tz = a\bar{z}\ \forall z$ (complex-antilinear).

(3) $\exists\, s > 0 : \langle Tw, Tz\rangle = s\langle w, z\rangle\ \forall z, w \in \mathbb{C}$.

Proof. (1) $\Rightarrow$ (2): Let $a := T(1)$. Since multiplication by any nonzero complex number is itself angle-preserving, $S := a^{-1}T$ is angle-preserving, with $S(1) = 1$.

Let $r :=$ the value with $S(i) = ri$: since $\langle 1, i\rangle = 0$, angle preservation forces $\langle S(1), S(i)\rangle = \langle 1, S(i)\rangle = 0$, so $S(i)$ is purely imaginary, $S(i) = ri$ for some $r \in \mathbb{R}$ (and $r \neq 0$ since S is invertible). By $\mathbb{R}$-linearity, $S(1 + i) = 1 + ri$ and $S(1 - i) = 1 - ri$. Since $\langle 1 + i, 1 - i\rangle = 1 \cdot 1 + 1 \cdot (-1) = 0$, angle preservation gives

$$0 = \langle S(1 + i), S(1 - i)\rangle = \langle 1 + ri,\, 1 - ri\rangle = 1 - r^2,$$

so $r = \pm 1$.

If $r = 1$: $S(i) = i$, so $S(x + iy) = xS(1) + yS(i) = x + iy = z$, i.e. $S = \mathrm{id}$, and $Tz = aSz = az$.

If $r = -1$: $S(i) = -i$, so $S(x + iy) = x - iy = \bar{z}$, and $Tz = aSz = a\bar{z}$.

(2) $\Rightarrow$ (3): If $Tz = az$, $\langle Tz, Tw\rangle = \operatorname{Re}(az\overline{aw}) = |a|^2 \operatorname{Re}(z\bar{w}) = |a|^2\langle z, w\rangle$, so $s = |a|^2$. If $Tz = a\bar{z}$, $\langle Tz, Tw\rangle = \operatorname{Re}(a\bar{z}\,\overline{a\bar{w}}) = |a|^2 \operatorname{Re}(w\bar{z}) = |a|^2\langle z, w\rangle$ as well.

(3) $\Rightarrow$ (1): $|Tz| = \sqrt{s}|z|$, $|Tw| = \sqrt{s}|w|$ (take $z = w$ in (3)). Then

$$|w||z|\langle Tw, Tz\rangle = |w||z| \cdot s\langle w, z\rangle = \big(\sqrt{s}|w|\big)\big(\sqrt{s}|z|\big)\langle w, z\rangle = |Tw||Tz|\langle w, z\rangle. \qquad \blacksquare$$

2 Conformal Maps

Setup: piecewise smooth arcs

A curve is written $(x(t), y(t))$, $t \in [a, b]$, in real notation, or $z(t) = x(t) + iy(t)$ in complex notation. Its **tangent** is nonzero when $(x'(t), y'(t)) \neq (0, 0)$, i.e. $z'(t) \neq 0$. The curve is **piecewise smooth** if there is a finite set $S \subset [a, b]$ with $z'(t) \neq 0$ for all $t \in [a, b] \setminus S$.

Let f be real differentiable at z_0, and let $z(t)$ be a curve with $z(t_0) = z_0$, $z'(t_0) \neq 0$. If T is the differential of f at z_0 (i.e. T is $\mathbb{R}$-linear and $|f(z_0 + h) - f(z_0) - T(h)|/|h| \to 0$ as $h \to 0$), then the chain rule gives

$$\big(f(z(t))\big)'\Big|_{t_0} = T\big(z'(t_0)\big).$$

T is called the **tangent map** (or **differential**) of f at z_0.

Definition (angle-preserving at a point)

f is **angle-preserving at** z_0 if its tangent map at z_0 is angle-preserving.

Lemma

The following are equivalent:

(1) f is angle-preserving at z_0.

(2) Either $\dfrac{\partial f}{\partial \bar{z}} = 0$ and $\dfrac{\partial f}{\partial z} \neq 0$ at z_0, or $\dfrac{\partial f}{\partial z} = 0$ and $\dfrac{\partial f}{\partial \bar{z}} \neq 0$ at z_0.

(3) The tangent map T satisfies $\langle Tw, Tz\rangle = s\langle w, z\rangle$ for some $s > 0$.

Proof. This is just a restatement of the previous Lemma, applied to T, using the Wirtinger decomposition $Tz = \dfrac{\partial f}{\partial z}(z_0) \cdot z + \dfrac{\partial f}{\partial \bar{z}}(z_0) \cdot \bar{z}$: T is complex-linear exactly when $\partial f/\partial \bar{z} = 0$, and complex-antilinear exactly when $\partial f/\partial z = 0$. □

Theorem

Let f be continuously real differentiable on a region D. Then f is angle-preserving at every $z \in D$ if and only if

$$f \in \mathcal{A}(D),\ f'(z) \neq 0\ \forall z \in D \qquad \text{or} \qquad \bar{f} \in \mathcal{A}(D),\ (\bar{f})'(z) \neq 0\ \forall z \in D.$$

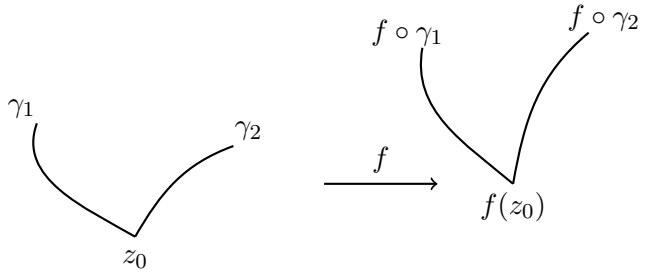
Proof. The "if" direction is immediate from the Lemma. For "only if": by the Lemma, at every $z \in D$ either $\partial f/\partial \bar{z} = 0$, $\partial f/\partial z \neq 0$, or $\partial f/\partial z = 0$, $\partial f/\partial \bar{z} \neq 0$. In either case exactly one of $\partial f/\partial z$, $\partial f/\partial \bar{z}$ vanishes, so the sum $\partial f/\partial z + \partial f/\partial \bar{z}$ never vanishes, and the ratio

$$\frac{\dfrac{\partial f}{\partial z} - \dfrac{\partial f}{\partial \bar{z}}}{\dfrac{\partial f}{\partial z} + \dfrac{\partial f}{\partial \bar{z}}}$$

is well defined at every point of D, continuous (as f is continuously differentiable), and takes only the values $+1$ (where $\partial f/\partial \bar{z} = 0$) or -1 (where $\partial f/\partial z = 0$). By connectedness of D, this ratio is constant.

If the constant is 1: $\partial f/\partial \bar{z} \equiv 0$ on D, i.e. $f \in \mathcal{A}(D)$. If the constant is -1: $\partial f/\partial z \equiv 0$ on D, i.e. $\bar{f} \in \mathcal{A}(D)$. ■

3 Geometric Meaning



If γ_1 and γ_2 intersect at angle ρ (at a point z_0 where f is angle-preserving), then $f \circ \gamma_1$ and $f \circ \gamma_2$ intersect at angle ρ as well – this is the geometric content of conformality.