Complex Analysis

Assignment 4, due October 22

Problem 1 of 5. Let $f(z) = e^z$. Fix $m \neq 0$, let $L_m = \{x + imx : x \in \mathbb{R}\}$ be the line through 0 of slope m, and let $S_m = f(L_m)$.

(1) Write S_m in polar form $r = r(\theta)$, where θ is a real, continuous parameter. Show that S_m is a logarithmic spiral, that f is one-to-one on L_m, and that S_m meets every ray from 0 infinitely many times.

(2) Show that S_m meets every ray from 0 at the same angle, and compute this angle in terms of m. Deduce that S_m also meets every circle $|w| = r$ at one and the same angle.

(3) Find all z such that $f(z) = -1 + i$, and describe the set $f^{-1}\big(\{|w| = 1\}\big)$.

Problem 2 of 5. Problem 1, page 73 of *Ahlfors*.

Problem 3 of 5.

(1) Find all zeros of $\sinh z$ and of $\cosh z$.

(2) Show that for every $z \in \mathbb{C}$ the equation $\sinh w = z$ has a solution w, and that all solutions are given by

$$w = \log\big(z + \sqrt{z^2 + 1}\big),$$

where both the logarithm and the square root take all their values. (This is the multivalued function $\operatorname{arsinh} z$.)

(3) Find all solutions of $\sinh w = 2$ and of $\sinh w = i$. Explain why the second equation has fewer solutions than the general formula of part (2) suggests.

Problem 4 of 5. Problem 3, page 73 of *Ahlfors*.

Problem 5 of 5 (*Conformality breaks at* 1). Let $J(z) = \frac{1}{2}\big(z + \frac{1}{z}\big)$ be the Joukowski map.

(1) Show that $J(z) - 1 = \dfrac{(z-1)^2}{2z}$, and deduce that J is analytic with $J'(z) \neq 0$ for $z \neq 0, \pm 1$, but $J'(1) = 0$. Show that angles at $z = 1$ are doubled: if two smooth arcs through 1 meet at the angle α, then their images under J meet at 1 at the angle 2α.

(2) Let $0 < \varepsilon < 1$ and let $C = \{z : |z + \varepsilon| = 1 + \varepsilon\}$ be the circle with centre $-\varepsilon$ through 1. Show that $J(C)$ has a cusp at $w = 1$: the two arcs of $J(C)$ leaving $w = 1$ are tangent to each other (to the negative real axis), even though C itself is smooth at 1.

(3) Show that J is one-to-one on C, so that $J(C)$ is a simple closed curve (an "airfoil" with a sharp trailing edge).

Hint: $J(z_1) = J(z_2)$ if and only if $z_1 = z_2$ or $z_1 z_2 = 1$; find the image of C under $z \mapsto 1/z$.