Complex Analysis

Assignment 3, due October 8

Problem 1 of 5. Call a polynomial p *Hurwitz-stable* if all of its roots have negative real part. Using the Gauss–Lucas Theorem, show that a non-constant derivative of a Hurwitz-stable polynomial, of any order, is again Hurwitz-stable.

Problem 2 of 5. Let $R(z) = P(z)/Q(z)$ be a rational function with $\deg P < \deg Q$, and suppose Q has $n = \deg Q$ distinct (simple) zeros $z_1, \ldots, z_n$.

(1) Show that the partial fraction decomposition of R takes the form

$$R(z) = \sum_{k=1}^{n} \frac{c_k}{z - z_k}, \qquad c_k = \frac{P(z_k)}{Q'(z_k)}.$$

(2) Apply part (1) to $R(z) = \dfrac{1}{z^n - 1}$ to show that

$$\frac{1}{z^n - 1} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\zeta^k}{z - \zeta^k}, \qquad \zeta = e^{2\pi i/n}.$$

Problem 3 of 5. Let $\sum_{n=1}^{\infty} z_n$ be a convergent series of complex numbers.

(1) Let $0 < \alpha < \pi/2$. Show that if for all n, $|\operatorname{Arg} z_n| < \alpha$, then the series $\sum_{n=1}^{\infty} z_n$ converges absolutely.

(2) Does the conclusion of the previous part hold if $\alpha = \pi/2$?
Hint: Consider a sequence whose real part decays much faster than its imaginary part.

(3) Assume that for all n, $\Re z_n \geq 0$ and that the series $\sum_{n=1}^{\infty} z_n^2$ also converges. Show that this series (i.e., $\sum_{n=1}^{\infty} z_n^2$) converges absolutely.
Hint: Consider $\Re(z_n^2)$.

Problem 4 of 5. Let $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ be two power series with the radii of convergence R_1 and R_2 respectively. Prove that

(1) The radius of convergence R of the series $\sum_{n=0}^{\infty} (a_n + b_n) z^n$ satisfies $R \geq \min(R_1, R_2)$ and that equality holds if $R_1 \neq R_2$.

(2) The radius of convergence R of the series $\sum_{n=0}^{\infty} (a_n b_n) z^n$ satisfies $R \geq R_1 R_2$.

Problem 5 of 5. Determine the radius of convergence of each of the following series:

(1) $\sum_{n=1}^{\infty} \left(\frac{7n^4 + 2n^3}{3n^4 + 2n} \right) z^n$

(2) $\sum_{n=1}^{\infty} \left(n^2 + a^n \right) z^n$, $a \in \mathbb{C}$

(3) $\sum_{n=1}^{\infty} (\sin n)\, z^n$