Complex Analysis

Assignment 1, due September 24

Problem 1 of 5. Let $s > 0$ be a fixed positive real number, and consider the equation

$$\left(z^2 + \overline{z}^2\right)(1 - r) + 2z\overline{z}(1 + r) = s$$

in the unknown $z \in \mathbb{C}$.

(1) Assume that r is a fixed real number. What shape is described by the set of solutions z? Your answer may depend on r and s.

(2) Now let r be a fixed *complex* number, not necessarily real, while $s > 0$ remains a fixed positive real number. What is the set of solutions $z \in \mathbb{C}$ of the same equation in this case? Your answer may depend on r and s.

Problem 2 of 5. Let A be a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, and let M_w denote the matrix (from class) representing multiplication by $w \in \mathbb{C}$.

(1) Show that A is *complex linear*, i.e.

$$A(\lambda z) = \lambda A(z), \quad \forall \lambda \in \mathbb{C},\ z \in \mathbb{C},$$

if and only if A has a matrix M_w for some $w \in \mathbb{C}$. Show that in this case $A(z) = wz$.

(2) Show that A is *complex anti-linear*, i.e.

$$A(\lambda z) = \overline{\lambda} A(z), \quad \forall \lambda \in \mathbb{C},\ z \in \mathbb{C},$$

if and only if A has a matrix $M_w \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for some $w \in \mathbb{C}$. Show that in this case $A(z) = w\overline{z}$.

Problem 3 of 5. Let $p > 1$ and $q = \frac{p}{p-1}$.

(1) Let $a \geq 0$ and $b \geq 0$. Show that the rectangle $\{(x, y) : 0 \leq x \leq a;\ 0 \leq y \leq b\}$ is contained in the union of subgraphs

$$\left\{(x, y) : 0 \leq x \leq a;\ 0 \leq y \leq x^{p-1}\right\} \cup \left\{(x, y) : 0 \leq y \leq b;\ 0 \leq x \leq y^{1/(p-1)}\right\}.$$

Hint: Sketch the curve $y = x^{p-1}$ inside the rectangle (equivalently $x = y^{1/(p-1)}$, since $t \mapsto t^{p-1}$ is a strictly increasing bijection of $[0, \infty)$ onto itself for $p > 1$). Use this monotonicity to argue that every point (x, y) of the rectangle satisfies either $y \leq x^{p-1}$ or $x \leq y^{1/(p-1)}$.

(2) Use the previous inclusion and integration to prove *Young's inequality*:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where $a \geq 0$ and $b \geq 0$.

(3) Use Young's inequality to show that if $a_j \geq 0$, $b_j \geq 0$ and

$$\sum_{j=1}^{n} a_j^p = \sum_{j=1}^{n} b_j^q = 1,$$

then

$$\sum_{j=1}^{n} a_j b_j \leq 1.$$

(4) Prove *Hölder inequality*:

$$\left|\sum_{j=1}^{n} z_j w_j\right| \leq \left(\sum_{j=1}^{n} |z_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |w_j|^q\right)^{\frac{1}{p}},$$

where z_j and w_j are arbitrary complex numbers. When $p = q = 2$, this is called *Cauchy inequality.*

Hint: Take $a_j = \frac{|z_j|}{\left(\sum_{j=1}^{n} |z_j|^p\right)^{\frac{1}{p}}}$ and $b_j = \frac{|w_j|}{\left(\sum_{j=1}^{n} |w_j|^q\right)^{\frac{1}{q}}}$ and apply the previous part. Be sure to deal with the case when one of the denominators is zero.

Problem 4 of 5. Let z_1, z_2, z_3 be three complex numbers not lying on the same line, so that they are the vertices of a triangle.

(1) For $t \in [0, 1]$, show that the point $z = (1-t)z_i + tz_j$ lies on the segment from z_i to z_j, and that it divides this segment in the ratio $t : (1-t)$, i.e. $|z_i - z| : |z - z_j| = t : (1-t)$. Using this, write down the midpoints of the three sides of the triangle in terms of z_1, z_2, z_3.

(2) Let $w = \dfrac{z_1 + z_2 + z_3}{3}$. Recall (or look up) the definition of a *median* of a triangle, and state it here. Show that w lies on all three medians of the triangle. State the theorem you have just proved.

(3) Generalize: instead of the midpoints, choose a point strictly between the two endpoints of each side of the triangle (i.e. on the side, but distinct from either vertex), and join each of these points to the opposite vertex. Denote by $t_1, t_2, t_3 \in (0, 1)$ the ratios in which your three chosen points divide the sides, and write the points themselves in terms of $z_1, z_2, z_3, t_1, t_2, t_3$ using part (a). Under what condition on t_1, t_2, t_3 do the resulting three lines meet at a single point? (This result is known as *Ceva's Theorem.*)

Problem 5 of 5. Consider the stereographic projection between $\mathbb{C}$ and the unit sphere $S \subset \mathbb{R}^3$, using the normalization from class. A circle on S is centered at the center of the sphere precisely when it is a *great circle*, i.e. the intersection of S with a plane through the origin. Give a complete description of the curves in $\mathbb{C} \cup \{\infty\}$ that arise as the images of great circles under this correspondence.